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Module maps on duals of Banach algebras and topological centre problems ☆

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Abstract

We study various spaces of module maps on the dual of a Banach algebra A, and relate them to topological centres. We introduce an auxiliary topological centre $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond$ for the left quotient Banach algebra $\langle A^*A\rangle^*$ of A^{**}. Our results indicate that $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond$ is indispensable for investigating properties of module maps over A^* and for understanding some asymmetry phenomena in topological centre problems as well as the interrelationships between different Arens irregularity properties. For the class of Banach algebras of type (M) introduced recently by the authors, we show that strong Arens irregularity can be expressed both in terms of automatic normality of A^{**}-module maps on A^* and through certain commutation relations. This in particular generalizes the earlier work on group algebras by Ghahramani and McClure (1992) [13] and by Ghahramani and Lau (1997) [12]. We link a module map property over A^* to the space $WAP(A)$ of weakly almost periodic functionals on A, generalizing a result by Lau and Ülger (1996) [34] for Banach algebras with a bounded approximate identity. We also show that for a locally compact quantum group $\mathbb{G}$, the quotient strong Arens irregularity of $L_1(\mathbb{G})$ can be obtained from that of $M(\mathbb{G})$ and can be characterized via the canonical $C_0(\mathbb{G})$-module structure on $LUC(\mathbb{G})^*$.

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1. Introduction

Let A be a Banach algebra. We use $\Box$ and $\Diamond$, respectively, to denote the left and the right Arens products on A^{**}, and $\mathfrak{Z}_t(A^{**},\Box)$ and $\mathfrak{Z}_t(A^{**},\Diamond)$, respectively, to denote the left and the right topological centres of A^{**} (see Section 2 for the definitions). The dual space A^* is canonically a Banach A-bimodule and a Banach left $(A^{**},\Box)$-module. The present paper has been motivated by the study of the following two properties of A.

(i) A is left strongly Arens irregular; that is, $\mathfrak{Z}_t(A^{**},\Box)=A$.
(ii) Every bounded left $(A^{**},\Box)$-module map on A^* is w^*-w^* continuous.

For a locally compact group G, it was shown by Lau and Losert [31] that the convolution group algebra $L_1(G)$ is (left) strongly Arens irregular. Properties (i) and (ii) were considered together in [13, Theorem 1.8], where Ghahramani and McClure showed implicitly that (i) implies (ii) for $A=L_1(G)$. In [38, Satz 3.7.7], Neufang proved that (i) and (ii) are equivalent if $A=L_1(G)$ and G is metrizable. He further proved in [39, Theorem 2.3] that, for Banach algebras A of type (MF) (i.e., A has the Mazur property of level κ and A^* has the left A^{**} factorization property of level κ for some cardinal $\kappa\geqslant\aleph_0$) such as $L_1(G)$ with G non-compact, property (i) holds, and property (ii) can even be strengthened to "every left $(A^{**},\Box)$-module map on A^* is automatically bounded and w^*-w^* continuous". For general Banach algebras, (i) always implies (ii) (cf. [39, Proposition 2.6]). Among other results, we shall show that for the class of Banach algebras of type (M) introduced and studied recently by the authors [21], properties (i) and (ii) are equivalent. We note that A^* is also a Banach right $(A^{**},\Diamond)$-module via the action $(f,m)\longmapsto f\Diamond m$, and all the results mentioned above have their right-hand side versions.

In this paper, we shall focus on the Banach algebra $B_A(X)$ of bounded right A-module maps on a Banach right A-module X, and hence most results are stated in their right-hand side versions even if X is an A-bimodule. When a right A-module X is the dual space of a given Banach space, we let $B_A^\sigma(X)$ denote the subalgebra of $B_A(X)$ consisting of w^*-w^* continuous maps in $B_A(X)$. We use $B_{A^{**}}(A^*)$ to denote the Banach algebra of bounded right $(A^{**},\Diamond)$-module maps on A^*. Then we have

$$B_A^\sigma(A^*)=B_{A^{**}}^\sigma(A^*)\subseteq B_{A^{**}}(A^*)\subseteq B_A(A^*). \tag{1.1}$$

Combining the right-hand side versions of the above results by Lau and Losert [31] and by Ghahramani and McClure [13], we can conclude that $B_{L_1(G)}^\sigma(L_\infty(G))=B_{L_1(G)^{**}}(L_\infty(G))$ for all locally compact groups G. In [26, Theorem 2], Lau showed that $B_{L_1(G)}^\sigma(L_\infty(G))=B_{L_1(G)}(L_\infty(G))$ precisely when G is compact. For more general Banach algebras, we shall study under what circumstances the equalities in (1.1) hold, and how these equalities are related to topological centre problems.

The paper is organized as follows. We start Section 2 with notation conventions followed by introducing an auxiliary topological centre $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond$ for $(\langle A^*A\rangle^*,\Box)$, the canonical quotient Banach algebra of $(A^{**},\Box)$. We show that $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond$ happens to be the hidden piece that is responsible for some asymmetry phenomena occurring in topological centre problems as observed in [20,34]. Results in the paper indicate that $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond$ is indispensable for the comparison between the algebras $B_A^\sigma(A^*)$, $B_{A^{**}}(A^*)$, and $B_A(A^*)$, and for the study of the interrelationships between different Arens irregularity (respectively, Arens regularity) properties. We close this section by summarizing some general results on module maps.

In Section 3, we study various spaces of module maps on A^*, and relate them to topological centres. We show that some of the automatic normality properties of module maps define new concepts between strong Arens irregularity and quotient strong Arens irregularity, the two notions introduced, respectively, by Dales and Lau [6] and by the authors [20]. More characterizations of module map properties over A^* are obtained for Banach algebras A of type (M). We also characterize the left faithfulness of the algebras $(A^{**}, \Box)$ and $(\langle A^*A\rangle^*, \Box)$ through topological centres. We note that there is another notion of "maximal" Arens irregularity in the literature, the notion of extreme non-Arens regularity introduced by Granirer [14]. The reader is referred to [19] and references therein for results on this type of Arens irregularity and its relationship to strong Arens irregularity.

In Section 4, we discuss commutation relations for module maps on A^*. We obtain several bicommutant theorems, which in particular improve and extend the commutant theorem [12, Theorem 5.1] on $L_1(G)$ by Ghahramani and Lau to all Banach algebras of type (M). For this class of Banach algebras, we show that left and right strong Arens irregularities are in fact equivalent to certain commutation and double commutation relations between module maps. In particular, we show that a unital weakly sequentially complete Banach algebra A (e.g., $A = A(G)$ with G compact) is right strongly Arens irregular if and only if we have the bicommutant theorem $A^{cc} = A$ for the canonical embedding $A \hookrightarrow B_A(A^*)$.

We introduced in [20] the Banach algebra $\langle A^*A\rangle_R^*$, which is a subspace of $(\langle A^*A\rangle^*, \Box)$ but equipped with a distinct multiplication. In Section 5, we consider the corresponding Banach algebra of module maps on A^*. This new Banach algebra structure is used in particular to characterize the introversion of $\langle A^*A\rangle$ in A^* and the equality $\langle A^*A\rangle = \langle AA^*A\rangle$. We further relate a module map property over A^* to the algebra $\langle A^*A\rangle_R^*$ and the space $WAP(A)$ of weakly almost periodic functionals on A, in particular generalizing [34, Theorem 3.6] by Lau and Ülger for Banach algebras with a bounded approximate identity.

For a locally compact quantum group $\mathbb{G}$, let $L_1(\mathbb{G})$ and $M(\mathbb{G})$ be the quantum group algebra and quantum measure algebra of $\mathbb{G}$, respectively. Let $LUC(\mathbb{G}) = \langle L_1(\mathbb{G})^* \star L_1(\mathbb{G})\rangle$ be the space of left uniformly continuous functionals on $L_1(\mathbb{G})$. In Section 6, we obtain a natural completely isometric $M(\mathbb{G})$-module isomorphism $LUC(\mathbb{G}) \cong \langle M(\mathbb{G})^* \star L_1(\mathbb{G})\rangle$, and characterize the quotient strong Arens irregularity of $L_1(\mathbb{G})$ in terms of the canonical $C_0(\mathbb{G})$-module structure on $LUC(\mathbb{G})^*$. We prove that if $M(\mathbb{G})$ is quotient strongly Arens irregular, then $L_1(\mathbb{G})$ is also quotient strongly Arens irregular. This in particular shows that for all amenable locally compact groups G, the strong Arens irregularity of the Fourier–Stieltjes algebra $B(G)$ implies that of the Fourier algebra $A(G)$. In the subsequent work [22], we will investigate module maps over locally compact quantum groups through a general Banach algebra approach as used in the present paper.

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2. Definitions and preliminary results

Let $\mathcal{B}$ be an algebra equipped with a topology such that $\mathcal{B}$ is a topological linear space. Suppose that $\mathcal{B}$ is also a right topological semigroup under the multiplication. That is, for any fixed $y \in \mathcal{B}$, the map $x \longmapsto xy$ is continuous on $\mathcal{B}$ (cf. [2]). In this case, the topological centre $\mathfrak{Z}_t(\mathcal{B})$ of $\mathcal{B}$ is defined to be the set of all $y \in \mathcal{B}$ such that the map $x \longmapsto yx$ is continuous on $\mathcal{B}$. When

$\mathcal{B}$ is a left topological semigroup under the multiplication, the topological centre $\mathfrak{Z}_t(\mathcal{B})$ of $\mathcal{B}$ is defined analogously.

Throughout this paper, A denotes a Banach algebra with a faithful multiplication. As is well known, on the bidual A^{**} of A, there are two Banach algebra multiplications called, respectively, the left and the right Arens products, each extending the multiplication on A. By definition, the left Arens product $\Box$ is induced by the left A-module structure on A. That is, for $m, n \in A^{**}$, $f \in A^*$, and $a, b \in A$, we have

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle, \qquad \langle n \Box f, a\rangle = \langle n, f \cdot a\rangle, \quad \text{and} \quad \langle m \Box n, f\rangle = \langle m, n \Box f\rangle.$$

The right Arens product $\Diamond$ is defined by considering A as a right A-module. More precisely, we have

$$\langle b, a \cdot f\rangle = \langle ba, f\rangle, \qquad \langle a, f \Diamond n\rangle = \langle a \cdot f, n\rangle, \quad \text{and}$$
$$\langle f, m \Diamond n\rangle = \langle f \Diamond m, n\rangle \quad \bigl(a, b \in A,\ f \in A^*,\ m, n \in A^{**}\bigr).$$

It is known that $m \Box n = w^*\text{-}\lim_\alpha w^*\text{-}\lim_\beta a_\alpha b_\beta$ and $m \Diamond n = w^*\text{-}\lim_\beta w^*\text{-}\lim_\alpha a_\alpha b_\beta$ whenever (a_α) and (b_β) are nets in A converging, respectively, to m and n in the weak*-topology on A^{**}. The Banach algebra A is said to be *Arens regular* if $\Box$ and $\Diamond$ coincide on A^{**}. Since the multiplication on a von Neumann algebra is separately w^*-w^* continuous, every C^*-algebra is Arens regular. Hence, by [4, Corollaries 6.3 and 6.4], every operator algebra and every quotient algebra thereof are Arens regular (see [8] for more information on the structure of the bidual of an operator algebra).

Under the weak*-topology, $(A^{**}, \Box)$ is a right topological semigroup and $(A^{**}, \Diamond)$ is a left topological semigroup. By definition, we have

$$\mathfrak{Z}_t\bigl(A^{**}, \Box\bigr) = \bigl\{m \in A^{**}\colon \text{ the map } n \longmapsto m \Box n \text{ is } w^*\text{-}w^* \text{ continuous on } A^{**}\bigr\}$$

and

$$\mathfrak{Z}_t\bigl(A^{**}, \Diamond\bigr) = \bigl\{m \in A^{**}\colon \text{ the map } n \longmapsto n \Diamond m \text{ is } w^*\text{-}w^* \text{ continuous on } A^{**}\bigr\}.$$

The algebras $\mathfrak{Z}_t(A^{**}, \Box)$ and $\mathfrak{Z}_t(A^{**}, \Diamond)$ are called, respectively, the left and the right topological centres of A^{**}. It is easy to see that

$$\mathfrak{Z}_t\bigl(A^{**}, \Box\bigr) = \bigl\{m \in A^{**}\colon\ m \Box n = m \Diamond n \text{ for all } n \in A^{**}\bigr\}$$

and

$$\mathfrak{Z}_t\bigl(A^{**}, \Diamond\bigr) = \bigl\{m \in A^{**}\colon\ n \Diamond m = n \Box m \text{ for all } n \in A^{**}\bigr\}.$$

Therefore, A is Arens regular if and only if $\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Diamond) = A^{**}$.

For a set X in a Banach space, we use $\langle X\rangle$ to denote the closed linear span of X in the space. By Cohen's factorization theorem, $\langle A^*A\rangle = A^*A$ if A has a bounded right approximate identity (BRAI), and $\langle AA^*\rangle = AA^*$ if A has a bounded left approximate identity (BLAI). The canonical quotient map $A^{**} \longrightarrow \langle A^*A\rangle^*$, $m \longmapsto m|_{\langle A^*A\rangle}$ yields a Banach algebra multiplication on $\langle A^*A\rangle^*$ (also denoted by $\Box$) such that we have the isometric Banach algebra identification

$$\big(\langle A^*A\rangle^*, \Box\big) \cong \big(A^{**}, \Box\big)/\langle A^*A\rangle^{\perp}.$$

Under the weak*-topology, $(\langle A^*A\rangle^*, \Box)$ is also a right topological semigroup. In the rest of the paper, $(\langle A^*A\rangle^*, \Box)$ and its topological centre are simply denoted by $\langle A^*A\rangle^*$ and $\mathfrak{Z}_t(\langle A^*A\rangle^*)$, respectively. The Banach algebra $(\langle AA^*\rangle^*, \Diamond)$ (or $\langle AA^*\rangle^*$ for short) is defined analogously as a quotient algebra of $(A^{**}, \Diamond)$.

Recall that the weak* operator topology (w^*ot) on $B(A^*)$ is the locally convex topology determined by the seminorms $T \in B(A^*) \longmapsto |\langle T(f), a\rangle|$ ($a \in A$, $f \in A^*$), and $B(A^*)$ is a dual Banach space via the isometric Banach space identification $B(A^*) \cong (A^* \otimes^\gamma A)^*$, where $\otimes^\gamma$ is the projective tensor product. Clearly, w^*-convergence in $B(A^*)$ implies w^*ot-convergence, and the converse holds on bounded subsets of $B(A^*)$. Then $B_A(A^*)$ is w^*ot-closed and hence w^*-closed in $B(A^*)$. For each $T \in B_A(A^*)$, the map $B_A(A^*) \longrightarrow B_A(A^*)$, $S \longmapsto S \circ T$ is w^*ot-w^*ot continuous and also w^*-w^* continuous. Thus $(B_A(A^*), w^*ot)$ and $(B_A(A^*), w^*)$ are both right topological semigroups. Let

$$B_A^{w^*ot}\big(A^*\big) = \big\{T \in B_A\big(A^*\big):\ S \longmapsto T \circ S \text{ is } w^*ot\text{-}w^*ot \text{ continuous on } B_A\big(A^*\big)\big\} \tag{2.1}$$

and

$$B_A^{w^*}\big(A^*\big) = \big\{T \in B_A\big(A^*\big):\ S \longmapsto T \circ S \text{ is } w^*\text{-}w^* \text{ continuous on } B_A\big(A^*\big)\big\}. \tag{2.2}$$

Then $B_A^{w^*ot}(A^*)$ and $B_A^{w^*}(A^*)$ are, respectively, the topological centres of $(B_A(A^*), w^*ot)$ and $(B_A(A^*), w^*)$, and we have $B_A^{w^*ot}(A^*) \subseteq B_A^{w^*}(A^*)$ by the Krein–Šmulian theorem.

For m in A^{**} or $\langle A^*A\rangle^*$ and f in A^*, let $m_L(f) \in A^*$ be defined by

$$m_L(f)(a) = \langle m \,\Box\, f, a\rangle = \langle m, f \cdot a\rangle \quad (a \in A). \tag{2.3}$$

Then $m_L(f) \in \langle A^*A\rangle$ if $f \in \langle A^*A\rangle$. Let $\Phi : \langle A^*A\rangle^* \longrightarrow B_A(A^*)$ be the map $m \longmapsto m_L$. Note that Φ is just the adjoint of the right A-module map $A^* \otimes^\gamma A \longrightarrow \langle A^*A\rangle$, $f \otimes a \longmapsto f \cdot a$. Then Φ is a w^*-w^* continuous, contractive, and injective algebra homomorphism, and we have

$$\Phi\big(\langle A^*A\rangle^*\big) = \big\{m_L:\ m \in \langle A^*A\rangle^*\big\} = \big\{m_L:\ m \in A^{**}\big\}. \tag{2.4}$$

We call Φ the *canonical representation* of $\langle A^*A\rangle^*$ on A^*. If A has a BRAI, then $\langle A^*A\rangle^*$ has an identity. In this case, Φ is surjective (since $T = T^*(E)_L$ for all $T \in B_A(A^*)$, where E is a right identity of $(A^{**}, \Box)$), and is in fact a w^*-w^*ot (and hence w^*-w^*) homeomorphism, and the w^*ot and the weak*-topology coincide on $B_A(A^*)$. Furthermore, Φ is isometric if A has a contractive BRAI. Conversely, if Φ is surjective, then $\langle A^*A\rangle^*$ has an identity, which implies that A has a BRAI if $\langle A^2\rangle = A$ (cf. [16, Theorem 4(ii)]).

Let $RM(A)$ be the right multiplier algebra of A (with opposite composition as the multiplication). It is clear that $RM(A) \longrightarrow B_A^{\sigma}(A^*)$, $\mu \longmapsto \mu^*$ is an isometric algebra isomorphism. Then we have

$$RM(A) \cong B_A^{\sigma}\big(A^*\big) = \big\{T \in B_A\big(A^*\big):\ T^*(A) \subseteq A\big\}. \tag{2.5}$$

If A has a BRAI, then $B_A^\sigma(A^*) \subseteq \Phi(\langle A^*A\rangle^*)$. Conversely, if $B_A^\sigma(A^*) \subseteq \Phi(\langle A^*A\rangle^*)$, then $\langle A^*A\rangle^*$ is unital, since $B_A^\sigma(A^*)$ contains the identity of $B_A(A^*)$ and Φ is an injective algebra homomorphism.

By the definition of $\mathfrak{Z}_t(A^{**}, \diamond)$, we have

$$B_{A^{**}}(A^*) = B_A^{r,w^*}(A^*) := \{T \in B_A(A^*)\colon\ T^*(A) \subseteq \mathfrak{Z}_t(A^{**}, \diamond)\}. \tag{2.6}$$

Corresponding to (2.3), for m in A^{**} or $\langle AA^*\rangle^*$ and f in A^*, we let $m_R(f) \in A^*$ be defined by

$$m_R(f)(a) = \langle a, f \diamond m\rangle = \langle a \cdot f, m\rangle \quad (a \in A). \tag{2.7}$$

Then we obtain

$$B_A^{r,w^*}(A^*) = \{T \in B_A(A^*)\colon\ A^{**} \longrightarrow B(A^*),\ m \longmapsto T \circ m_R \text{ is } w^*\text{-}w^* \text{ continuous}\}. \tag{2.8}$$

Replacing $\mathfrak{Z}_t(A^{**}, \diamond)$ in (2.6) by $\mathfrak{Z}_t(A^{**}, \Box)$, we can define the space

$$B_A^{l,w^*}(A^*) = \{T \in B_A(A^*)\colon\ T^*(A) \subseteq \mathfrak{Z}_t(A^{**}, \Box)\}. \tag{2.9}$$

In fact, comparing with (2.8), we can show that

$$B_A^{l,w^*}(A^*) = \{T \in B_A(A^*)\colon\ A^{**} \longrightarrow B(A^*),\ m \longmapsto T \circ m_L \text{ is } w^*\text{-}w^* \text{ continuous}\}. \tag{2.10}$$

Therefore, we have

$$\begin{aligned} B_A^\sigma(A^*) &\subseteq B_A^{w^*ot}(A^*) \subseteq B_A^{w^*}(A^*) \subseteq B_A^{l,w^*}(A^*) \\ &\subseteq \{T \in B_A(A^*)\colon\ T^*(A)|_{\langle A^*A\rangle} \subseteq \mathfrak{Z}_t(\langle A^*A\rangle^*)\}. \end{aligned} \tag{2.11}$$

We note that w^*-w^* continuity in (2.8) and (2.10) can be replaced by w^*-w^*ot continuity.

It is known from [20, Corollary 3(i)] that

$$\mathfrak{Z}_t(\langle A^*A\rangle^*) = \{m \in \langle A^*A\rangle^*\colon\ A \cdot m \subseteq \mathfrak{Z}_t(A^{**}, \Box)\}. \tag{2.12}$$

Hence, we also have $\Phi(\mathfrak{Z}_t(\langle A^*A\rangle^*)) \subseteq B_A^{l,w^*}(A^*)$. As mentioned above, if A has a BRAI, then the map $\Phi : \langle A^*A\rangle^* \longrightarrow B_A(A^*)$ is a w^*-w^* homeomorphism; in this case, we obtain

$$\begin{aligned} \Phi(\mathfrak{Z}_t(\langle A^*A\rangle^*)) &= B_A^{w^*ot}(A^*) = B_A^{w^*}(A^*) = B_A^{l,w^*}(A^*) \\ &= \{T \in B_A(A^*)\colon\ T^*(A)|_{\langle A^*A\rangle} \subseteq \mathfrak{Z}_t(\langle A^*A\rangle^*)\}, \end{aligned}$$

and, in particular, $B_A^{l,w^*}(A^*)$ is a Banach algebra. Clearly, the module map spaces $B_A^{l,w^*}(A^*)$ and $B_A^{r,w^*}(A^*)$ $(= B_{A^{**}}(A^*))$ are closely related to topological centres. They will be used to study several Arens irregularity properties in the paper.

It is seen that $\mathfrak{Z}_t(A^{**}, \Box)$ and $\mathfrak{Z}_t(A^{**}, \diamond)$ both appear in the study of (the one-sided) right A-module maps on A^*. We observe that one natural A-submodule of $\langle A^*A\rangle^*$ has not been considered in the discussions so far. This A-bimodule relates $\langle A^*A\rangle^*$ to $\mathfrak{Z}_t(A^{**}, \diamond)$, playing a similar role as $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ does in (2.12) where $\langle A^*A\rangle^*$ is linked to $\mathfrak{Z}_t(A^{**},\Box)$. The results obtained in the paper indicate that this Banach A-bimodule is exactly the missing piece in the study of Arens irregularity, without which some asymmetries occur in topological centre problems (cf. [20, Remark 28] and [34, Remark 5.2]). This A-bimodule shall be used to describe the pre-image of $B_{A^{**}}(A^*)$ under Φ, to compare the algebras $B_A^\sigma(A^*)$, $B_{A^{**}}(A^*)$, and $B_A(A^*)$, and to study further interrelationships between various topological centre problems and properties of module maps on A^* (see the results presented in this and the next sections).

Before giving the definition, let us first note that $A^*\Diamond\langle A^{**}A\rangle\subseteq\langle A^*A\rangle$ holds, since we have

$$f\Diamond(n\cdot a)=f\Diamond(n\Diamond a)=(f\Diamond n)\Diamond a=(f\Diamond n)\cdot a\quad\bigl(a\in A,\ f\in A^*,\ n\in A^{**}\bigr).$$

For $m\in\langle A^*A\rangle^*$, let $\widetilde{m}\in A^{**}$ be any extension of m. Then, for $n\in\langle A^{**}A\rangle$ and $p\in A^{**}$, we see that

$$n\Diamond m=n\Diamond\widetilde{m}\quad\text{and}\quad p\Box m=p\Box\widetilde{m}\tag{2.13}$$

are well-defined elements of A^{**}. Therefore, for $m\in\langle A^*A\rangle^*$, we obtain that $A\cdot m\subseteq\mathfrak{Z}_t(A^{**},\Diamond)$ if and only if $n\Diamond m=n\Box m$ in A^{**} for all $n\in\langle A^{**}A\rangle$.

Definition 2.1. Let A be a Banach algebra. The *auxiliary topological centre* of $\langle A^*A\rangle^*$ is defined by

$$\mathfrak{Z}_t\bigl(\langle A^*A\rangle^*\bigr)_\Diamond=\bigl\{m\in\langle A^*A\rangle^*:\ n\Diamond m=n\Box m\text{ in }A^{**}\text{ for all }n\in\langle A^{**}A\rangle\bigr\}.$$

The auxiliary topological centre $\mathfrak{Z}_t(\langle AA^*\rangle^*)_\Box$ of $\langle AA^*\rangle^*$ is defined similarly.

Obviously, $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond$ is just the right topological centre $\mathfrak{Z}_t(A^{**},\Diamond)$ of A^{**} if A is unital. In general, $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond$ is a Banach A-submodule of $\langle A^*A\rangle^*$. Comparing with (2.11) and (2.12), we have

$$\mathfrak{Z}_t\bigl(\langle A^*A\rangle^*\bigr)_\Diamond=\bigl\{m\in\langle A^*A\rangle^*:\ A\cdot m\subseteq\mathfrak{Z}_t\bigl(A^{**},\Diamond\bigr)\bigr\}\tag{2.14}$$

and

$$B_A^\sigma\bigl(A^*\bigr)\subseteq B_{A^{**}}\bigl(A^*\bigr)=B_A^{r,w^*}\bigl(A^*\bigr)\subseteq\bigl\{T\in B_A\bigl(A^*\bigr):\ T^*(A)|_{\langle A^*A\rangle}\subseteq\mathfrak{Z}_t\bigl(\langle A^*A\rangle^*\bigr)_\Diamond\bigr\}.\tag{2.15}$$

Therefore, $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond=\mathfrak{Z}_t(\langle A^*A\rangle^*)$ if $\mathfrak{Z}_t(A^{**},\Box)=\mathfrak{Z}_t(A^{**},\Diamond)$. In general, $\Phi(\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond)\subseteq B_{A^{**}}(A^*)$, and $\Phi(\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond)=B_{A^{**}}(A^*)=\{T\in B_A(A^*):\ T^*(A)|_{\langle A^*A\rangle}\subseteq\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond\}$ if A has a BRAI.

In the sequel, we use the symbol $B_A^{r,w^*}(A^*)$ for the algebra $B_{A^{**}}(A^*)$ when we want to show the parallel between $B_A^{l,w^*}(A^*)$ and $B_A^{r,w^*}(A^*)$. The proposition below on $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ and $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond$ is clearly true.

Proposition 2.2. *Let A be a Banach algebra. Then the following assertions hold.*

(i) *The canonical quotient map $\varphi : A^{**} \longrightarrow \langle A^*A\rangle^*$ satisfies*

$$\varphi\big(\mathfrak{Z}_t\big(A^{**},\Box\big)\big) \subseteq \mathfrak{Z}_t\big(\big\langle A^*A\big\rangle^*\big) \quad \text{and} \quad \varphi\big(\mathfrak{Z}_t\big(A^{**},\Diamond\big)\big) \subseteq \mathfrak{Z}_t\big(\big\langle A^*A\big\rangle^*\big)_\Diamond.$$

(ii) *The canonical representation $\Phi : \langle A^*A\rangle^* \longrightarrow B_A(A^*)$ satisfies*

$$\Phi\big(\mathfrak{Z}_t\big(\big\langle A^*A\big\rangle^*\big)\big) \subseteq B_A^{l,w^*}\big(A^*\big) \quad \text{and} \quad \Phi\big(\mathfrak{Z}_t\big(\big\langle A^*A\big\rangle^*\big)_\Diamond\big) \subseteq B_A^{r,w^*}\big(A^*\big).$$

(iii) $\mathfrak{Z}_t(A^{**},\Box) = \mathfrak{Z}_t(A^{**},\Diamond) \Longrightarrow B_A^{l,w^*}(A^*) = B_A^{r,w^*}(A^*) \Longrightarrow \mathfrak{Z}_t(\langle A^*A\rangle^*) = \mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond$.

(iv) $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond$ *is a subalgebra of* $(\langle A^*A\rangle^*,\Box)$ *in the following cases*:
   (a) $\mathfrak{Z}_t(A^{**},\Diamond) = A$;
   (b) *A has a BRAI*;
   (c) $\langle A^2\rangle = A$ *and* $A\cdot\mathfrak{Z}_t(A^{**},\Diamond) \subseteq A$;
   (d) $\mathfrak{Z}_t(A^{**},\Diamond)\Box A^{**} \subseteq \mathfrak{Z}_t(A^{**},\Diamond)$.

For an easy comparison, we record the following results on topological centres from [20].

Proposition 2.3. *(See [20].) Let A be a Banach algebra. Then the following assertions hold.*

(i) $\Phi(\mathfrak{Z}_t(\langle A^*A\rangle^*)) \subseteq B_A^\sigma(A^*) \Longleftrightarrow A\cdot\mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq A$.

(ii) *If $\langle A^2\rangle = A$, then we have*

$$A\cdot\mathfrak{Z}_t\big(A^{**},\Box\big) \subseteq A \quad \Longleftrightarrow \quad A\cdot\mathfrak{Z}_t\big(\big\langle A^*A\big\rangle^*\big) \subseteq A$$

and

$$\mathfrak{Z}_t\big(A^{**},\Diamond\big)\cdot A \subseteq A \quad \Longleftrightarrow \quad \mathfrak{Z}_t\big(\big\langle AA^*\big\rangle^*\big)\cdot A \subseteq A.$$

(iii) *If $A^2 = A$, in particular, if A has a BRAI or a BLAI, then we have*

$$A\cdot\mathfrak{Z}_t\big(A^{**},\Box\big) = A\cdot\mathfrak{Z}_t\big(\big\langle A^*A\big\rangle^*\big) \quad \text{and} \quad \mathfrak{Z}_t\big(A^{**},\Diamond\big)\cdot A = \mathfrak{Z}_t\big(\big\langle AA^*\big\rangle^*\big)\cdot A.$$

In Proposition 2.3, $\mathfrak{Z}_t(A^{**},\Box)$ and $\mathfrak{Z}_t(A^{**},\Diamond)$ are on the "wrong" sides, i.e., the left (respectively, right) topological centre stays on the right-hand (respectively, left-hand) side. The proposition below shows that they can be pulled to the "correct" sides by using the auxiliary topological centres.

Proposition 2.4. *Let A be a Banach algebra. Then the following assertions hold.*

(i) $\Phi(\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond) \subseteq B_A^\sigma(A^*) \Longleftrightarrow A\cdot\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond \subseteq A$.

(ii) *If $\langle A^2\rangle = A$, then we have*

$$\mathfrak{Z}_t\big(A^{**},\Box\big)\cdot A \subseteq A \quad \Longleftrightarrow \quad \mathfrak{Z}_t\big(\big\langle AA^*\big\rangle^*\big)_\Box\cdot A \subseteq A$$

and

$$A\cdot\mathfrak{Z}_t\big(A^{**},\Diamond\big) \subseteq A \quad \Longleftrightarrow \quad A\cdot\mathfrak{Z}_t\big(\big\langle A^*A\big\rangle^*\big)_\Diamond \subseteq A.$$

(iii) *If $A^2 = A$, in particular, if A has a BRAI or a BLAI, then we have*

$$\mathfrak{Z}_t(A^{**}, \Box) \cdot A = \mathfrak{Z}_t(\langle AA^* \rangle^*)_\Box \cdot A \quad \textit{and} \quad A \cdot \mathfrak{Z}_t(A^{**}, \Diamond) = A \cdot \mathfrak{Z}_t(\langle A^*A \rangle^*)_\Diamond.$$

Proof. (i) is obvious. Clearly, $a \cdot m = a \cdot m|_{\langle A^*A \rangle}$ for all $a \in A$ and $m \in A^{**}$, and $m|_{\langle A^*A \rangle} \in \mathfrak{Z}_t(\langle A^*A \rangle^*)_\Diamond$ if $m \in \mathfrak{Z}_t(A^{**}, \Diamond)$. Then we obtain $A \cdot \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A \cdot \mathfrak{Z}_t(\langle A^*A \rangle^*)_\Diamond \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$. Similarly, we have $\mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq \mathfrak{Z}_t(\langle AA^* \rangle^*)_\Box \cdot A \subseteq \mathfrak{Z}_t(A^{**}, \Box)$. Therefore, (ii) and (iii) hold. □

Proposition 2.5. *Let A be a Banach algebra. Then the following assertions hold.*

(i) $\mathfrak{Z}_t(\langle A^*A \rangle^*) = \langle A^*A \rangle^* \iff A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \Box)$, *which is equivalent to* $A \cdot \langle A^*A \rangle^* \subseteq \mathfrak{Z}_t(\langle A^*A \rangle^*)$ *if A satisfies* $\langle A^2 \rangle = A$.
(ii) $\mathfrak{Z}_t(\langle A^*A \rangle^*)_\Diamond = \langle A^*A \rangle^* \iff A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$, *which is equivalent to* $A \cdot \langle A^*A \rangle^* \subseteq \mathfrak{Z}_t(\langle A^*A \rangle^*)_\Diamond$ *if A satisfies* $\langle A^2 \rangle = A$.

Proof. We prove (i); assertion (ii) can be shown similarly.

The first equivalence follows from (2.12). Clearly, if $\mathfrak{Z}_t(\langle A^*A \rangle^*) = \langle A^*A \rangle^*$, then $A \cdot \langle A^*A \rangle^* \subseteq \mathfrak{Z}_t(\langle A^*A \rangle^*)$. Conversely, suppose that $\langle A^2 \rangle = A$ and $A \cdot \langle A^*A \rangle^* \subseteq \mathfrak{Z}_t(\langle A^*A \rangle^*)$. Let $a, b \in A$ and $m \in A^{**}$, and let $p = m|_{\langle A^*A \rangle} \in \langle A^*A \rangle^*$. Then $b \cdot p \in \mathfrak{Z}_t(\langle A^*A \rangle^*)$. Thus, by (2.12) again, we have

$$(ab) \cdot m = a \cdot (b \cdot m) = a \cdot (b \cdot p) \in A \cdot \mathfrak{Z}_t(\langle A^*A \rangle^*) \subseteq \mathfrak{Z}_t(A^{**}, \Box).$$

That is, $A^2 \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \Box)$. Therefore, $A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \Box)$ since $\langle A^2 \rangle = A$. □

For convenience, we summarize the facts discussed and implied above in the following two propositions. Most of these facts can be found in some form in the existing literature, but some of them were stated only for certain classes of Banach algebras. See, for example, [1,3,5,6,16,20,21,26,27,30,34]. The reader is also referred to [15] for a systematic study of left (respectively, right) Banach modules of the form $\langle V^*A \rangle$ (respectively, $\langle AV^* \rangle$), where V is a left (respectively, right) Banach A-module.

Proposition 2.6. *Let A be a Banach algebra satisfying $\langle A^2 \rangle = A$. Then the following statements are equivalent*:

(i) *A has a BRAI*;
(ii) $\Phi : \langle A^*A \rangle^* \longrightarrow B_A(A^*)$ *is surjective*;
(iii) $\langle A^*A \rangle^*$ *is unital*;
(iv) $B_A^\sigma(A^*) \subseteq \Phi(\langle A^*A \rangle^*)$;
(v) $B_A^\sigma(A^*) \subseteq \Phi(\mathfrak{Z}_t(\langle A^*A \rangle^*))$ (*respectively,* $B_A^\sigma(A^*) \subseteq \Phi(\mathfrak{Z}_t(\langle A^*A \rangle^*)_\Diamond)$);
(vi) $\Phi(\mathfrak{Z}_t(\langle A^*A \rangle^*)) = B_A^{l,w^*}(A^*)$ (*respectively,* $\Phi(\mathfrak{Z}_t(\langle A^*A \rangle^*)_\Diamond) = B_A^{r,w^*}(A^*)$).

Proposition 2.7. *Let A be a Banach algebra with a BRAI. Then $\Phi : \langle A^*A \rangle^* \longrightarrow B_A(A^*)$ is a w^*-w^* homeomorphism which is isometric if A has a contractive BRAI, and the following assertions hold.*

(i) $B_A^{w^*ot}(A^*) = B_A^{w^*}(A^*) = B_A^{l,w^*}(A^*)$.
(ii) $B_A^{r,w^*}(A^*) = \Phi(\mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamondsuit) = \{m_L\colon\ m \in A^{**}$ *and* $A \cdot m \subseteq \mathfrak{Z}_t(A^{**}, \diamondsuit)\}$.
(iii) $B_A^{l,w^*}(A^*) = \Phi(\mathfrak{Z}_t(\langle A^*A\rangle^*)) = \{m_L\colon\ m \in A^{**}$ *and* $A \cdot m \subseteq \mathfrak{Z}_t(A^{**}, \Box)\}$.
(iv) *If A is unital, then we have* $B_A^\sigma(A^*) \cong A$, $B_A(A^*) \cong (A^{**}, \Box)$, $B_A^{r,w^*}(A^*) \cong \mathfrak{Z}_t(A^{**}, \diamondsuit)$, *and* $B_A^{l,w^*}(A^*) \cong \mathfrak{Z}_t(A^{**}, \Box)$; *in this case, A is Arens regular* $\Longleftrightarrow B_A(A^*) = B_A^{l,w^*}(A^*) \Longleftrightarrow B_A(A^*) = B_A^{r,w^*}(A^*)$.

Therefore, for every unital C^*-algebra A, we have

$$B_A^\sigma(A^*) \cong A \quad \text{and} \quad B_A^{r,w^*}(A^*) = B_A^{l,w^*}(A^*) = B_A(A^*) \cong A^{**} \cong \pi_u(A)'',$$

where $\pi_u(A)''$ is the enveloping von Neumann algebra of A.

It is known that for a general *left introverted* subspace X of A^* (that is, X is a closed right A-submodule of A^* satisfying $A^{**} \Box X \subseteq X$), the Banach algebra $(X^*, \Box)$ and its topological centre $\mathfrak{Z}_t(X^*)$ are also defined. In this case, the map $\Psi_X : X^* \longrightarrow B_A(X)$, $m \longmapsto m_L$ is a contractive algebra homomorphism, where m_L is given as before by $\langle m_L(f), a\rangle = \langle m, f \cdot a\rangle$ ($m \in X^*$, $f \in X$, $a \in A$). The proposition below can be shown easily. We omit the proof.

Proposition 2.8. *Let A be a Banach algebra and let X be a left introverted subspace of A^*. Then the following assertions hold.*

(i) *Ψ_X is injective $\Longleftrightarrow [X = \langle X \cdot A\rangle] \Longleftrightarrow$ the product $\Box$ on X^* is right faithful, which is equivalent to $X \subseteq \langle A^*A\rangle$ if A has a BRAI.*
(ii) *Ψ_X is surjective $\Longleftrightarrow X^*$ is right unital.*
(iii) *Suppose that A is w^*-dense in X^*. Then Ψ_X is bijective $\Longleftrightarrow X^*$ is left unital $\Longleftrightarrow X^*$ is unital.*

Remark 2.9. Note that $T(\langle A^*A\rangle) \subseteq \langle A^*A\rangle$ for all $T \in B_A(A^*)$. Let $\Pi : B_A(A^*) \longrightarrow B_A(\langle A^*A\rangle)$ be the map $T \longmapsto T|_{\langle A^*A\rangle}$. Then Π is a contractive algebra homomorphism, and is injective if $\langle A^2\rangle = A$. By Propositions 2.6 and 2.8 (with $X = \langle A^*A\rangle$), the map Π is bijective if A has a BRAI. However, the converse is not true. For example, if $A = A(\mathbb{F}_2)$ is the Fourier algebra of the free group $\mathbb{F}_2$ of two generators, then A has no BRAI, but $\Pi : B_A(A^*) \longrightarrow B_A(\langle A^*A\rangle)$ is bijective, since $\langle A^*A\rangle$ is the reduced group C^*-algebra $C_\lambda^*(\mathbb{F}_2)$ of $\mathbb{F}_2$ and, for every $S \in B_A(\langle A^*A\rangle)$, we have $S = \Pi(T)$ with $T = (S^*|_A)^* \in B_A(A^*)$.

In fact, the above arguments can be applied to a general locally compact quantum group $\mathbb{G}$, which show that if $\mathbb{G}$ is co-amenable or compact, then Π yields an isometric algebra isomorphism $B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) \cong B_{L_1(\mathbb{G})}(LUC(\mathbb{G}))$. Recall that a locally compact quantum group $\mathbb{G}$ is *co-amenable* if the quantum group algebra $L_1(\mathbb{G})$ has a BAI, and is *compact* if the C^*-algebra $C_0(\mathbb{G})$ is unital. See Section 6 and references therein for more information on locally compact quantum groups.

Analogously, *right introverted* subspaces of A^* and their topological centres can be considered. In the present paper, however, we shall focus on the case where $X = \langle A^*A\rangle$. The corresponding results with $\langle A^*A\rangle$ replaced by $\langle AA^*\rangle$ also hold. Some of the results given in the paper do not require the faithfulness of the multiplication on A, and many of them have their *cb*-versions. In particular, if A is a completely contractive Banach algebra, then the canonical representation Φ induces a completely contractive injection $\langle A^*A\rangle^* \longrightarrow CB_A(A^*)$, where

$CB_A(A^*)$ is the algebra of completely bounded maps in $B_A(A^*)$. Therefore, we have the following immediate consequence of Proposition 2.6.

Corollary 2.10. *Let A be a completely contractive Banach algebra with a BRAI. Then we have*

$$B_A(A^*) = CB_A(A^*) \quad \text{and} \quad RM(A^*) = RM_{cb}(A^*),$$

where $RM_{cb}(A^*)$ denotes the completely bounded right multiplier algebra of A.

We note that recently Viktor Losert showed that $RM(A(SL(2,\mathbb{R}))) = RM_{cb}(A(SL(2,\mathbb{R})))$ though the Fourier algebra $A(SL(2,\mathbb{R}))$ does not have a BRAI.

3. Module maps and topological centres

The results in Section 2 show in particular that the comparison between $B_A^\sigma(A^*)$, $B_{A^{**}}(A^*)$, and $B_A^{l,w^*}(A^*)$ does describe Arens irregularity of a Banach algebra A. In other words, the automatic normality of certain module maps is intrinsically related to topological centre problems. We start this section with a corollary of Propositions 2.2–2.5 (cf. (2.4) and (2.5)). As in Section 2, we shall use the symbol $B_A^{r,w^*}(A^*)$ for the algebra $B_{A^{**}}(A^*)$ when a result contains both $B_A^{l,w^*}(A^*)$ and $B_{A^{**}}(A^*)$.

Corollary 3.1. *Let A be a Banach algebra. Then the following assertions hold.*

(i) *If $B_A^{r,w^*}(A^*) = B_A^\sigma(A^*)$, then $A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond \subseteq A$.*
(ii) *If $B_A(A^*) = B_A^{r,w^*}(A^*)$, then $A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \diamond)$.*
(iii) *If $B_A^{l,w^*}(A^*) = B_A^\sigma(A^*)$, then $A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq A$.*
(iv) *If $B_A(A^*) = B_A^{l,w^*}(A^*)$, then $A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \square)$.*
(v) *If $B_A(A^*) = B_A^\sigma(A^*)$, then $A \cdot A^{**} \subseteq A$.*

We shall show that under the mild condition that "$\langle A^2\rangle = A$", the converse of each of (i)–(v) holds. Note that the predual A of any Hopf–von Neumann algebra $\mathcal{M}$ satisfies this condition, where the multiplication on A is the pre-adjoint of the co-multiplication on $\mathcal{M}$ and hence is a (complete) quotient map. In the sequel, the right multiplier algebra $RM(A)$ and the algebra $\langle A^*A\rangle^*$ are identified with their canonical images in $B_A(A^*)$.

Theorem 3.2. *Let A be a Banach algebra satisfying $\langle A^2\rangle = A$ (e.g., A is the predual of a Hopf–von Neumann algebra). Then the statements* (i)–(iv) *in each of* (I)–(V) *are equivalent.*

(I) (i) $B_A^{r,w^*}(A^*) = B_A^\sigma(A^*)$.
(ii) $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond \subseteq RM(A)$.
(iii) $A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond \subseteq A$.
(iv) $A \cdot \mathfrak{Z}_t(A^{**}, \diamond) \subseteq A$.

(II) (i) $B_A(A^*) = B_A^{r,w^*}(A^*)$.
(ii) $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond = \langle A^*A\rangle^*$.
(iii) $A \cdot \langle A^*A\rangle^* \subseteq \mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond$.
(iv) $A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \diamond)$.

(III) (i) $B_A^{l,w^*}(A^*) = B_A^\sigma(A^*)$.
(ii) $\mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq RM(A)$.

(iii) $A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq A$.
(iv) $A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A$.

(IV) (i) $B_A(A^*) = B_A^{l,w^*}(A^*)$.
(ii) $\mathfrak{Z}_t(\langle A^*A\rangle^*) = \langle A^*A\rangle^*$.
(iii) $A \cdot \langle A^*A\rangle^* \subseteq \mathfrak{Z}_t(\langle A^*A\rangle^*)$.
(iv) $A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \Box)$.

(V) (i) $B_A(A^*) = B_A^{\sigma}(A^*)$.
(ii) $\langle A^*A\rangle^* \subseteq RM(A)$.
(iii) $A \cdot \langle A^*A\rangle^* \subseteq A$.
(iv) $A \cdot A^{**} \subseteq A$.

In addition, the inclusions in (I), (III), *and* (V) *can be replaced by the equalities if A has a BRAI.*

Proof. (I) Due to Proposition 2.4 and Corollary 3.1(i), we need only show that (iv) $\Longrightarrow$ (i).

Suppose that $A \cdot \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A$ and $T \in B_A^{r,w^*}(A^*)$. Let $a, b \in A$. By (2.6), we have $T^*(b) \in \mathfrak{Z}_t(A^{**}, \Diamond)$, and thus $T^*(ab) = a \cdot T^*(b) \in A \cdot \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A$. Hence, $T^*(A) = T^*(\langle A^2\rangle) \subseteq A$, and $T \in B_A^{\sigma}(A^*)$ (cf. (2.5)). Therefore, $B_A^{r,w^*}(A^*) = B_A^{\sigma}(A^*)$.

(II) Due to Proposition 2.5(ii) and Corollary 3.1(ii), we need only prove that (iv) $\Longrightarrow$ (i).

Suppose that $A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$ and $T \in B_A(A^*)$. Let $a, b \in A$. Then $T^*(ab) = a \cdot T^*(b) \in A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$. Hence, $T^*(A) = T^*(\langle A^2\rangle) \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$, and $T \in B_A^{r,w^*}(A^*)$ (cf. (2.6)). Therefore, we have $B_A(A^*) = B_A^{r,w^*}(A^*)$.

(III) The proof is similar to that of (I), where Proposition 2.4, Corollary 3.1(i), and (2.6) are replaced by Proposition 2.3, Corollary 3.1(iii), and (2.9), respectively.

(IV) The proof is similar to that of (II), where Proposition 2.5(ii), Corollary 3.1(ii), and (2.6) are replaced by Proposition 2.5(i), Corollary 3.1(iv), and (2.9), respectively.

(V) This is true by (I), (II) and Corollary 3.1(v).

The final assertion holds by Proposition 2.6. $\square$

Remark 3.3. (a) The equivalence between (ii)–(iv) in (III) has been proved in [20, Corollary 4 and Theorem 24]. (IV) generalizes the equivalence between b) and d) in [34, Theorem 3.6], which shows that if A has a bounded approximate identity (BAI), then $A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \Box)$ if and only if $\mathfrak{Z}_t(\langle A^*A\rangle^*) = \langle A^*A\rangle^*$ (cf. Proposition 2.6). Correspondingly, (V) generalizes [34, Corollary 3.7], which is equivalent to saying that for a Banach algebra A with a BAI, we have $A \cdot A^{**} \subseteq A$ if and only if $\langle A^*A\rangle^* = RM(A)$.

(b) Recall that a Banach algebra A is left strongly Arens irregular (LSAI) if $\mathfrak{Z}_t(A^{**}, \Box) = A$, a concept introduced and studied by Dales and Lau [6]. Right strong Arens irregularity (RSAI) is defined similarly. The algebra A is strongly Arens irregular (SAI) if it is both LSAI and RSAI. In [20], we say that A is left quotient strongly Arens irregular (LQ-SAI) if $\mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq RM(A)$, and left quotient Arens regular (LQ-AR) if $\mathfrak{Z}_t(\langle A^*A\rangle^*) = \langle A^*A\rangle^*$. Similarly, RQ-SAI and RQ-AR are defined via $\mathfrak{Z}_t(\langle AA^*\rangle^*)$ and $LM(A)$ (the left multiplier algebra of A). The algebra A is called quotient strongly Arens irregular (Q-SAI) if it is both LQ-SAI and RQ-SAI. Proposition 2.6 shows that, in a sense, A being LQ-SAI is opposite to A having a BRAI. In the spirit of this terminology, we say that A is *right-left quotient strongly Arens irregular* (RLQ-SAI) if $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond \subseteq RM(A)$, and *right-left quotient Arens regular* (RLQ-AR) if $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond = \langle A^*A\rangle^*$. Analogously, LRQ-SAI and LRQ-AR are defined through $LM(A)$ and

$\mathfrak{Z}_t(\langle AA^*\rangle^*)_\square$. Therefore, for Banach algebras A satisfying $\langle A^2\rangle = A$, the statements in Theorem 3.2(I)–(IV) indeed characterize RLQ-SAI, RLQ-AR, LQ-SAI, and LQ-AR, respectively.

(c) Let G be a locally compact group. It is known that $L_1(G)$ is Q-SAI and SAI (cf. [29,31]). The situation for the Fourier algebra $A(G)$ is very different. Firstly, since $A(G)$ is commutative, both topological centres of $A(G)^{**}$ are equal to the algebraic centre $\mathfrak{Z}(A(G)^{**})$ of $A(G)^{**}$ (with either Arens product). Secondly, on the one hand, for many amenable groups G, the Fourier algebra $A(G)$ is SAI (cf. [10,17,18,32,33]). On the other hand, as shown by Losert [35,36], both $A(\mathbb{F}_2)$ and $A(SU(3))$ are non-SAI, though $A(\mathbb{F}_2)$ is Q-SAI (by Theorem 3.2 and [28, Theorem 3.7]) and $SU(3)$ is compact.

Recently, Matthias Neufang and his Ph.D student Denis Poulin introduced and studied the strong left and right topological centres of A^{**}, which are defined by

$$\mathfrak{S}\mathfrak{Z}_t\big(A^{**},\square\big)_\ell := \big\{m \in A^{**}\colon\ \lambda_m = T^{**} \text{ for some } T \in B(A)\big\}$$

and

$$\mathfrak{S}\mathfrak{Z}_t\big(A^{**},\diamond\big)_r := \big\{m \in A^{**}\colon\ \rho^m = T^{**} \text{ for some } T \in B(A)\big\},$$

where λ_m and ρ^m are the maps on A^{**} given by $n \longmapsto m \,\square\, n$ and $n \longmapsto n \diamond m$, respectively (cf. [42]). By [21, Theorem 17], we have

$$\mathfrak{S}\mathfrak{Z}_t\big(A^{**},\square\big)_\ell = \mathfrak{Z}_t\big(A^{**},\square\big) \cap \big\{m \in A^{**}\colon\ m\cdot A \subseteq A\big\}; \tag{3.1}$$

$$\mathfrak{S}\mathfrak{Z}_t\big(A^{**},\diamond\big)_r = \mathfrak{Z}_t\big(A^{**},\diamond\big) \cap \big\{m \in A^{**}\colon\ A\cdot m \subseteq A\big\}. \tag{3.2}$$

We can define the other two strong topological centres of A^{**} and the strong topological centres of the two canonical quotient Banach algebras of A^{**} as follows:

$$\mathfrak{S}\mathfrak{Z}_t\big(A^{**},\square\big) := \mathfrak{Z}_t\big(A^{**},\square\big) \cap \big\{m \in A^{**}\colon\ A\cdot m \subseteq A\big\}; \tag{3.3}$$

$$\mathfrak{S}\mathfrak{Z}_t\big(A^{**},\diamond\big) := \mathfrak{Z}_t\big(A^{**},\diamond\big) \cap \big\{m \in A^{**}\colon\ m\cdot A \subseteq A\big\}; \tag{3.4}$$

$$\mathfrak{S}\mathfrak{Z}_t\big(\langle A^*A\rangle^*\big) := \big\{m \in \langle A^*A\rangle^*\colon\ A\cdot m \subseteq A\big\} \quad \text{and}$$
$$\mathfrak{S}\mathfrak{Z}_t\big(\langle AA^*\rangle^*\big) := \big\{m \in \langle AA^*\rangle^*\colon\ m\cdot A \subseteq A\big\}. \tag{3.5}$$

Due to (2.12), (2.14), and the corresponding $\langle AA^*\rangle$-versions, we have

$$\mathfrak{S}\mathfrak{Z}_t\big(\langle A^*A\rangle^*\big) \subseteq \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big) \cap \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big)_\diamond \quad \text{and}$$
$$\mathfrak{S}\mathfrak{Z}_t\big(\langle AA^*\rangle^*\big) \subseteq \mathfrak{Z}_t\big(\langle AA^*\rangle^*\big) \cap \mathfrak{Z}_t\big(\langle AA^*\rangle^*\big)_\square. \tag{3.6}$$

The algebra $\mathfrak{S}\mathfrak{Z}_t(\langle A^*A\rangle^*)$ in $\langle A^*A\rangle^*$ plays a role as $M(G)$ does in $LUC(G)^*$ (cf (6.7)). It is clear that

$$A \text{ is LQ-SAI} \quad \Longleftrightarrow \quad \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big) = \mathfrak{S}\mathfrak{Z}_t\big(\langle A^*A\rangle^*\big)$$

and

$$A \text{ is RQL-SAI} \quad \Longleftrightarrow \quad \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big)_\diamond = \mathfrak{S}\mathfrak{Z}_t\big(\langle A^*A\rangle^*\big).$$

Therefore, for Banach algebras A satisfying $\langle A^2\rangle = A$, Theorem 3.2 together with its $\langle AA^*\rangle$-version shows in particular the following relations between these twelve topological centres:

$$\mathfrak{Z}_t(A^{**},\Box) = \mathfrak{S}\mathfrak{Z}_t(A^{**},\Box) \iff \mathfrak{Z}_t(\langle A^*A\rangle^*) = \mathfrak{S}\mathfrak{Z}_t(\langle A^*A\rangle^*); \tag{3.7}$$

$$\mathfrak{Z}_t(A^{**},\Box) = \mathfrak{S}\mathfrak{Z}_t(A^{**},\Box)_\ell \iff \mathfrak{Z}_t(\langle AA^*\rangle^*)_\Box = \mathfrak{S}\mathfrak{Z}_t(\langle AA^*\rangle^*); \tag{3.8}$$

$$\mathfrak{Z}_t(A^{**},\Diamond) = \mathfrak{S}\mathfrak{Z}_t(A^{**},\Diamond) \iff \mathfrak{Z}_t(\langle AA^*\rangle^*) = \mathfrak{S}\mathfrak{Z}_t(\langle AA^*\rangle^*); \tag{3.9}$$

$$\mathfrak{Z}_t(A^{**},\Diamond) = \mathfrak{S}\mathfrak{Z}_t(A^{**},\Diamond)_r \iff \mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond = \mathfrak{S}\mathfrak{Z}_t(\langle A^*A\rangle^*). \tag{3.10}$$

The following result is evident by Propositions 2.2–2.4, and Theorem 3.2(I) and (III) (cf. (2.5), (2.6), and (2.9)). It shows that certain form of automatic normality yields properties between strong Arens irregularity and quotient strong Arens irregularity. More precisely, for a general Banach algebra A, the equality $B_A^{l,w^*}(A^*) = B_A^\sigma(A^*)$ defines a property between LSAI and LQ-SAI, and the equality $B_A^{r,w^*}(A^*) = B_A^\sigma(A^*)$ defines a property between RSAI and RLQ-SAI. The question of when LSAI and LQ-SAI are equivalent was investigated by the authors in [20].

Proposition 3.4. *Let A be a Banach algebra. Then, in each of* (I) *and* (II)*, we have* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii)*, and* (ii) $\Longleftrightarrow$ (iii) *if* $\langle A^2\rangle = A$.

(I) (i) *A is left strongly Arens irregular*;
 (ii) $B_A^{l,w^*}(A^*) = B_A^\sigma(A^*)$;
 (iii) *A is left quotient strongly Arens irregular.*

(II) (i) *A is right strongly Arens irregular*;
 (ii) $B_A^{r,w^*}(A^*) = B_A^\sigma(A^*)$;
 (iii) *A is right-left quotient strongly Arens irregular.*

The corollary below follows from Proposition 2.2 and Theorem 3.2, noticing that $B_{A^{**}}(A^*) = B_A^{r,w^*}(A^*)$.

Corollary 3.5. *Let A be a Banach algebra with a BRAI. If $\mathfrak{Z}_t(A^{**},\Box) = \mathfrak{Z}_t(A^{**},\Diamond)$, then we have*

(i) $B_{A^{**}}(A^*) = B_A^\sigma(A^*) \Longleftrightarrow \mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$;
(ii) $B_A(A^*) = B_{A^{**}}(A^*) \Longleftrightarrow \langle A^*A\rangle^* = \mathfrak{Z}_t(\langle A^*A\rangle^*)$.

Example 3.6. It is known that there exists a unital weakly sequentially complete Banach algebra A such that $\mathfrak{Z}_t(A^{**},\Box) = A \subsetneqq \mathfrak{Z}_t(A^{**},\Diamond)$ (cf. [20, page 636]). In this case, due to Proposition 2.7(iv), we have

$$\mathfrak{Z}_t(\langle A^*A\rangle^*) = \mathfrak{Z}_t(A^{**},\Box) = A = RM(A),$$
$$\text{but } B_{A^{**}}(A^*) = \mathfrak{Z}_t(A^{**},\Diamond) = \mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond \neq A = B_A^\sigma(A^*).$$

This shows that Corollary 3.5(i) does not hold in general if the topological centres $\mathfrak{Z}_t(A^{**},\Box)$ and $\mathfrak{Z}_t(A^{**},\Diamond)$ are different.

It is seen that various module map properties over A^* are closely related to the Banach modules $\mathfrak{Z}_t(A^{**},\Box)$, $\mathfrak{Z}_t(A^{**},\Diamond)$, $\mathfrak{Z}_t(\langle A^*A\rangle^*)$, $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond$, and $RM(A)$, and the relationship between $\mathfrak{Z}_t(A^{**},\Box)$ and $\mathfrak{Z}_t(A^{**},\Diamond)$. We shall further explore such connections for the classes of Banach algebras introduced recently by the authors [21]. Let us first recall the following definition.

Definition 3.7. (See [21].) Let A be a Banach algebra with a BAI. Then A is said to be of type (RM) if for every $\mu \in RM(A)$, there is a closed subalgebra B of A with a BAI such that

(I) $\mu|_B \in RM(B)$;
(II) $f|_B \in BB^*$ for all $f \in AA^*$;
(III) there is a family $\{B_i\}$ of closed right ideals in B satisfying (i) each B_i is weakly sequentially complete with a sequential BAI, (ii) for every i, there exists a left B_i-module projection from B onto B_i, and (iii) $\mu \in A$ if $\mu|_{B_i} \in B_i$ for all i.

Furthermore, the algebra A is said to be of type (RM^+) if the words "left" and "right" can be removed from condition (III). Similarly, Banach algebras of type (LM) (respectively, (LM^+)) are defined. The algebra A is said to be of type (M) if it is both of type (LM) and of type (RM), and of type (M^+) if it is both of type (LM^+) and of type (RM^+).

Obviously, a unital weakly sequentially complete Banach algebra is of type (M^+). For every locally compact group G, any convolution Beurling algebra $L(G,\omega)$ with $\omega \geqslant 1$ is of type (M^+), and so is the Fourier algebra $A(G)$ if G is amenable. Also, every separable quantum group algebra of a co-amenable locally compact quantum group is of type (M^+). It is known from [21, Theorem 18] that

$$\begin{aligned} \mathfrak{S}\mathfrak{Z}_t\big(A^{**},\Box\big)_\ell &= A \quad \text{if } A \text{ is of type } (LM) \quad \text{and} \\ \mathfrak{S}\mathfrak{Z}_t\big(A^{**},\Diamond\big)_r &= A \quad \text{if } A \text{ is of type } (RM). \end{aligned} \tag{3.11}$$

The reader is referred to [21] for results on Banach algebras from these classes.

We have the following generalization of [38, Satz 3.7.7], where the equivalence between (i) and (v) below was shown for the group algebra $L_1(G)$ with G metrizable.

Corollary 3.8. *Let A be a Banach algebra of type (RM). Then the following statements are equivalent*:

(i) $B_{A^{**}}(A^*) = B_A^\sigma(A^*)$;
(ii) $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond = RM(A)$;
(iii) $A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond = A$;
(iv) $A \cdot \mathfrak{Z}_t(A^{**},\Diamond) = A$;
(v) $\mathfrak{Z}_t(A^{**},\Diamond) = A$.

Proof. The equivalence between (i)–(iv) holds due to Theorem 3.2, and (ii) and (v) are equivalent by (3.10) and (3.11). $\Box$

Analogous to Corollary 3.8 on right $(A^{**}, \diamond)$-module maps, we can obtain the following characterizations of automatic normality of right A-module maps. In this situation, a natural subspace of A^{**} as discussed below plays a similar role as $\mathfrak{Z}_t(A^{**}, \diamond)$ does in Corollary 3.8.

Let us assume that A has a BAI. Then A^{**} has a *mixed identity* E (that is, E is a right identity of $(A^{**}, \Box)$ and a left identity of $(A^{**}, \diamond)$). Let $\mathcal{E}(A^{**})$ be the set of mixed identities of A^{**} and let

$$\Lambda_r\big(A^{**}\big) = \big\{m \in A^{**}\colon\ E \,\Box\, m = m \text{ for all } E \in \mathcal{E}\big(A^{**}\big)\big\}. \tag{3.12}$$

The space $\Lambda_r(A^{**})$ was denoted as $\mathcal{F}_1$ in [34], and as $\Lambda(G)$ in [33] where $A = A(G)$. Clearly, $\Lambda_r(A^{**})$ is a closed right ideal in $(A^{**}, \Box)$, and we have

$$A \subseteq \mathfrak{Z}_t\big(A^{**}, \diamond\big) \subseteq \Lambda_r\big(A^{**}\big) \subseteq A^{**}. \tag{3.13}$$

The equivalence between (iv) and (v) below was shown in [33, Proposition 5.4] for $A(G)$ with G amenable.

Corollary 3.9. *Let A be a Banach algebra of type (RM^+). Then the following statements are equivalent*:

(i) $B_A(A^*) = B_A^\sigma(A^*)$;
(ii) $\langle A^*A\rangle^* = RM(A)$;
(iii) $A \cdot \langle A^*A\rangle^* = A$;
(iv) $A \cdot A^{**} = A$;
(v) $A \cdot \Lambda_r(A^{**}) = A$;
(vi) $\Lambda_r(A^{**}) = A$.

Proof. The equivalence between (i)–(iv) follows from Theorem 3.2. Note that we have $A \cdot A^{**} \subseteq \Lambda_r(A^{**})$. Thus (vi) $\Longrightarrow$ (iv). It is obvious that (iv) $\Longrightarrow$ (v). Finally, (v) $\Longrightarrow$ (vi) holds by the definition of $\Lambda_r(A^{**})$ and [21, Proposition 27]. □

With the space $\Lambda_r(A^{**})$ involved in the discussion, we characterize below Banach algebras A with a BAI such that every bounded right A-module map on A^* is automatically a right $(A^{**}, \diamond)$-module map.

Corollary 3.10. *Let A be a Banach algebra with a BAI. Then the following statements are equivalent*:

(i) $B_A(A^*) = B_{A^{**}}(A^*)$;
(ii) $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond = \langle A^*A\rangle^*$;
(iii) $A \cdot \langle A^*A\rangle^* \subseteq \mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond$;
(iv) $A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \diamond)$;
(v) $A \cdot \Lambda_r(A^{**}) \subseteq \mathfrak{Z}_t(A^{**}, \diamond)$.

Proof. Due to Theorem 3.2(II), we only have to prove that (v) $\Longrightarrow$ (iv). To show this, we suppose that $A \cdot \Lambda_r(A^{**}) \subseteq \mathfrak{Z}_t(A^{**}, \diamond)$. Let $m \in A^{**}$ and $a \in A$. Since A has a BAI, we have $a = bc$

for some $b, c \in A$. Then $c \cdot m \in \Lambda_r(A^{**})$ and $a \cdot m = b \cdot (c \cdot m) \in A \cdot \Lambda_r(A^{**}) \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$. Therefore, we have $A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$. $\square$

Remark 3.11. Note that $A \cdot \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq \Lambda_r(A^{**})$. Hence, we have

$$\begin{aligned} \Lambda_r\big(A^{**}\big) = \mathfrak{Z}_t\big(A^{**}, \Diamond\big) \quad &\Longrightarrow \quad A \cdot \Lambda_r\big(A^{**}\big) \subseteq \mathfrak{Z}_t\big(A^{**}, \Diamond\big) \\ &\Longleftrightarrow \quad A \cdot \Lambda_r\big(A^{**}\big) = A \cdot \mathfrak{Z}_t\big(A^{**}, \Diamond\big). \end{aligned}$$

It is not clear for us when the reverse implication holds.

We consider below under what circumstances the equality $B_A^{w^*}(A^*) = B_A^{\sigma}(A^*)$ is equivalent to the left strong Arens irregularity of A.

Corollary 3.12. *Let A be a Banach algebra of type (RM). If $\mathfrak{Z}_t(A^{**}, \Box) \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$, then the following statements are equivalent*:

(i) $B_A^{w^*}(A^*) = B_A^{\sigma}(A^*)$;
(ii) $\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$;
(iii) $A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*) = A$;
(iv) $A \cdot \mathfrak{Z}_t(A^{**}, \Box) = A$;
(v) $\mathfrak{Z}_t(A^{**}, \Box) = A$.

Proof. In this case, we have $B_A^{w^*}(A^*) = B_A^{l,w^*}(A^*)$ (cf. Proposition 2.7(i)). The equivalence between (i)–(iv) then follows from Theorem 3.2, and (ii) and (v) are equivalent by [21, Theorem 25]. $\square$

Comparing with Corollary 3.8, we are unable to show in Corollary 3.12 that (v) is equivalent to the rest (i)–(iv) without the extra assumption that "$\mathfrak{Z}_t(A^{**}, \Box) \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$"; notice that, unlike the situation in Corollary 3.8(iv), the left topological centre $\mathfrak{Z}_t(A^{**}, \Box)$ in Corollary 3.12(iv) is on the "wrong" side.

The $\langle AA^*\rangle$-version of Corollary 3.8 together with Corollary 3.12 shows that for a Banach algebra A of type (M), the LSAI (i.e., $\mathfrak{Z}_t(A^{**}, \Box) = A$) is in fact intrinsically related to the LRQ-SAI (i.e., $\mathfrak{Z}_t(\langle AA^*\rangle^*)_\Box = LM(A)$) rather than the LQ-SAI (i.e., $\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$).

We have the following partial converse to the first half of Proposition 2.2(iii) (see also Example 3.6).

Corollary 3.13. *Let A be a Banach algebra of type (RM) that is LSAI. If $B_A^{l,w^*}(A^*) = B_A^{r,w^*}(A^*)$, then $\mathfrak{Z}_t(A^{**}, \Diamond) = \mathfrak{Z}_t(A^{**}, \Box) = A$ (i.e., A is SAI).*

Proof. Since A has a BAI and $\mathfrak{Z}_t(A^{**}, \Box) = A$, we have $B_A^{l,w^*}(A^*) = B_A^{\sigma}(A^*)$ by Theorem 3.2. Suppose that $B_A^{l,w^*}(A^*) = B_A^{r,w^*}(A^*)$. Then $B_{A^{**}}(A^*) = B_A^{\sigma}(A^*)$, and hence $\mathfrak{Z}_t(A^{**}, \Diamond) = A$ by Corollary 3.8. $\square$

We remark that we are unable to derive the conclusion of Corollary 3.13 when (RM) and LSAI are replaced by (LM) and RSAI, respectively (see the paragraph after Corollary 3.12).

It is easy to see that if A is Arens regular, then A is (left and right) quotient Arens regular (cf. Remark 3.3(b)). The converse is not true. For instance, if G is an infinite compact group, then $L_1(G)$ is quotient Arens regular but non-Arens regular. As left and right-left quotient Arens regularity properties are characterized in Theorem 3.2, we can also characterize Arens regularity of A in terms of module maps on A^* (see Proposition 2.7(iv) for the case where A is unital).

To see this, we recall that the weak operator topology (*wot*) on $B(A^*)$ is the locally convex topology determined by the seminorms $T \in B(A^*) \longmapsto |\langle Tx, n\rangle|$ $(x \in A^*, n \in A^{**})$. Let

$$B_A^{l,w}(A^*) = \{T \in B_A(A^*)\colon \text{ the map } A^{**} \longrightarrow B(A^*),\ m \longmapsto T \circ m_L \text{ is } w^*\text{-}wot \text{ continuous}\} \tag{3.14}$$

and

$$B_A^{r,w}(A^*) = \{T \in B_A(A^*)\colon \text{ the map } A^{**} \longrightarrow B(A^*),\ m \longmapsto T \circ m_R \text{ is } w^*\text{-}wot \text{ continuous}\}. \tag{3.15}$$

Then we have

$$B_A^{l,w}(A^*) = \{T \in B_A(A^*)\colon\ T^*(A^{**}) \subseteq \mathfrak{Z}_t(A^{**}, \Box)\} \subseteq B_A^{l,w^*}(A^*) \subseteq B_A(A^*)$$

and

$$B_A^{r,w}(A^*) = \{T \in B_A(A^*)\colon\ T^*(A^{**}) \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)\} \subseteq B_A^{r,w^*}(A^*) \subseteq B_A(A^*).$$

However, we may not have $B_A^\sigma(A^*) \subseteq B_A^{l,w}(A^*)$ or $B_A^\sigma(A^*) \subseteq B_A^{r,w}(A^*)$. In fact, these inclusions hold if and only if A is Arens regular. Comparing with Theorem 3.2(II) and (IV) on quotient Arens regularity, the proposition below on Arens regularity is clear.

Proposition 3.14. *Let A be a Banach algebra. Then the following statements are equivalent*:

(i) *A is Arens regular*;
(ii) $B_A(A^*) = B_A^{l,w}(A^*)$ (*respectively,* $B_A(A^*) = B_A^{r,w}(A^*)$);
(iii) $B_A^\sigma(A^*) \subseteq B_A^{l,w}(A^*)$ (*respectively,* $B_A^\sigma(A^*) \subseteq B_A^{r,w}(A^*)$);
(iv) $id \in B_A^{l,w}(A^*)$ (*respectively,* $id \in B_A^{r,w}(A^*)$).

Therefore, for every C^*-algebra A, we have $B_A(A^*) = B_A^{l,w}(A^*) = B_A^{r,w}(A^*)$.

Since A is faithful, we have $A \cap \langle AA^*\rangle^\perp = \{0\}$. It is known that $(A^{**}, \Box)$ is right faithful if and only if $A^* = \langle A^*A\rangle$, and $\langle A^*A\rangle^*$ is right faithful if and only if $\langle A^*A\rangle = \langle A^*A^2\rangle$ (cf. Proposition 2.8(i)). For a subset X of $\langle A^*A\rangle$, let $X^\perp_{\langle A^*A\rangle^*}$ denote the annihilator of X in $\langle A^*A\rangle^*$. Then we have $RM(A) \cap \langle AA^*A\rangle^\perp_{\langle A^*A\rangle^*} = \{0\}$. We also note that $\langle A^*A\rangle^* \Box \langle A^*A\rangle = A^{**} \Box \langle A^*A\rangle$, and $[\langle A^{**} \Box A^*\rangle = A^*] \Longrightarrow [\langle\langle A^*A\rangle^* \Box \langle A^*A\rangle\rangle = \langle A^*A\rangle]$. Therefore, we can derive from a brief calculation the following proposition on the algebras $(A^{**}, \Box)$ and $\langle A^*A\rangle^*$. Part of this proposition has been considered in [17] and [23] for commutative Banach algebras and quantum group algebras, respectively.

Proposition 3.15. *Let A be a Banach algebra. Then we have*

(i) $\mathfrak{Z}_t(A^{**}, \Box) \cap \langle AA^* \rangle^{\perp} = \langle A^{**} \Box A^* \rangle^{\perp}$;
(ii) $\mathfrak{Z}_t(\langle A^*A \rangle^*) \cap \langle AA^*A \rangle^{\perp}_{\langle A^*A \rangle^*} = \langle \langle A^*A \rangle^* \Box \langle A^*A \rangle \rangle^{\perp}_{\langle A^*A \rangle^*}$.

Therefore, the following assertions hold.

(a) $(A^{**}, \Box)$ *is left faithful* $\Longleftrightarrow \mathfrak{Z}_t(A^{**}, \Box) \cap \langle AA^* \rangle^{\perp} = \{0\}$.
(b) $\langle A^*A \rangle^*$ *is left faithful* $\Longleftrightarrow \mathfrak{Z}_t(\langle A^*A \rangle^*) \cap \langle AA^*A \rangle^{\perp}_{\langle A^*A \rangle^*} = \{0\}$.
(c) $\langle A^*A \rangle^*$ *is left faithful if* $(A^{**}, \Box)$ *is left faithful.*
(d) $(A^{**}, \Box)$ *is left faithful if A has a BRAI or is LSAI, or* $A^* = \langle AA^* \rangle$.
(e) $\langle A^*A \rangle^*$ *is left faithful if A has a BRAI or is LQ-SAI, or* $\langle A^*A \rangle = \langle AA^*A \rangle$.

Remark 3.16. (i) The converse of (c) is not true, since $A(\mathbb{F}_2)$ is Q-SAI, but $\langle A(\mathbb{F}_2)^{**} \Box A(\mathbb{F}_2)^* \rangle \neq A(\mathbb{F}_2)^*$ as shown by Losert [35]. Furthermore, the Banach algebra B_2 constructed in [19, Theorem 5] together with $B_2 \oplus L_1(G)$ of any non-SIN group G shows that none of the converses of (d) and (e) holds.

(ii) It follows from Proposition 3.15(i) that $\langle AA^* \rangle = \langle A^{**} \Box A^* \rangle$ if A is Arens regular. The converses is not true in general, since this equality always holds for every unital Banach algebra A. However, Ülger [44, Theorem 3.3] proved that if A is commutative, semisimple, weakly sequentially complete, and completely continuous with A^* a von Neumann algebra, then A is Arens regular if and only if $\langle AA^* \rangle = \langle A^{**} \Box A^* \rangle$. Therefore, we have $\langle A(\mathbb{F}_2)A(\mathbb{F}_2)^* \rangle \subsetneqq \langle A(\mathbb{F}_2)^{**} \Box A(\mathbb{F}_2)^* \rangle \subsetneqq A(\mathbb{F}_2)^*$.

(iii) It is easy to see that $\langle A^*A + A^{**} \Box (AA^*) \rangle^{\perp}$ is always a two-sided ideal in A^{**}. On the other hand, $\langle A^*A + AA^* \rangle^{\perp}$ is a two-sided ideal in A^{**} in the following two cases:

(1) $\langle AA^* \rangle \subseteq \langle A^*A \rangle$;
(2) $A^{**} \Box \langle AA^* \rangle \subseteq \langle AA^* \rangle$, which is true if $\langle AA^* \rangle = \langle A^{**} \Box A^* \rangle$ (e.g., A is Arens regular; see (ii) above).

For Banach algebras A satisfying $\langle A^2 \rangle = A$, we can show that

$$\langle A^*A + AA^* \rangle^{\perp} \text{ is a two-sided ideal in } A^{**} \quad \Longleftrightarrow \quad A^{**} \Box \langle AA^* \rangle \subseteq \langle A^*A + AA^* \rangle.$$

It is still open whether $\langle A^*A + AA^* \rangle^{\perp}$ is a two-sided ideal in A^{**} even for the case when A has a contractive BAI (see [15, Remark 4.38]).

Remark 3.17. For a Banach algebra A, let $\mathfrak{A} = (A^{**}, \Box)$. Then $B_{\mathfrak{A}}(\mathfrak{A}^*) \subseteq B_A(\mathfrak{A}^*)$. For $A = L_1(G)$, when studying whether the involution on A can be extended to an involution on $\mathfrak{A}$, Farhadi and Ghahramani [9] raised the following question:

$$\text{Do we have } T \in B_{\mathfrak{A}}(\mathfrak{A}^*) \text{ if } T \in B^{\sigma}_A(\mathfrak{A}^*) \text{ is surjective?} \tag{3.16}$$

In [40], Neufang proved that the answer to question (3.16) is negative for all $A = L_1(G)$ with G discrete, abelian, and countably infinite. We observe that the argument given in [40] shows that the answer to (3.16) is negative whenever A is a commutative Banach algebra with a BAI such that the map $m_L : A^* \longrightarrow A^*$ is bounded from below for some m in $A^{**} \setminus \mathfrak{Z}_t(A^{**}, \Box)$.

4. Commutation relations, bicommutant theorems, and strong Arens irregularity

Let A be a Banach algebra. For a subset X of $B(A^*)$, let X^c be the commutant of X in $B(A^*)$. We should point out that X^c is not w^*-closed in $B(A^*)$ in general. Let ${}_AB(A^*)$ (respectively, ${}_{A^{**}}B(A^*)$) be the Banach algebra of bounded *left* A-module (respectively, $(A^{**}, \Box)$-module) maps on A^*. Note that the embedding $A \longrightarrow B_A(A^*)$ via the canonical embedding $A \longrightarrow RM(A)$ is the same as the one obtained via the canonical embedding $A \longrightarrow \langle A^*A\rangle^*$, $a \longmapsto a|_{\langle A^*A\rangle}$. It is easy to see that $a_L(f) = a \cdot f$ for all $a \in A$ and $f \in A^*$. Therefore, we have $B_A^\sigma(A^*)^c \subseteq {}_AB(A^*)$.

Recall from (2.7) that for m in A^{**} or $\langle AA^*\rangle^*$ and f in A^*, $m_R(f) \in A^*$ is given by $m_R(f) = f \diamond m$. Then $m_R(f) \in \langle AA^*\rangle$ if $f \in \langle AA^*\rangle$ (that is, $\langle AA^*\rangle$ is right introverted in A^*). We note that the canonical representation $\langle AA^*\rangle^* \longrightarrow {}_AB(A^*)$, $m \longmapsto m_R$ is an anti-algebra homomorphism. Clearly,

$$\{m_R\colon m \in A^{**}\} = \{m_R\colon m \in \langle AA^*\rangle^*\} \subseteq {}_AB(A^*).$$

Thus we have $B_{A^{**}}(A^*)^c \subseteq B_A^\sigma(A^*)^c \subseteq {}_AB(A^*)$, and ${}_AB(A^*)^c \subseteq \{m_R\colon m \in A^{**}\}^c = B_{A^{**}}(A^*)$. Noticing that $a \cdot T(f) = T^*(a) \,\Box\, f$ $(a \in A,\ f \in A^*,\ T \in B_A(A^*))$, we can derive the following proposition.

Proposition 4.1. *Let A be a Banach algebra satisfying $\langle A^2\rangle = A$. Then we have*

$$B_A^\sigma(A^*) \subseteq {}_AB(A^*)^c \subseteq B_{A^{**}}(A^*) \subseteq {}_{A^{**}}B(A^*)^c = {}_AB^\sigma(A^*)^c = B_A(A^*).$$

We note that all the results established for $B_A(A^*)$ have their ${}_AB(A^*)$-versions. In particular, if A has a BLAI, then ${}_AB(A^*) = \{m_R\colon m \in A^{**}\}$ (cf. Proposition 2.6). In this case, ${}_AB(A^*)^c = B_{A^{**}}(A^*)$. Therefore, by Theorem 3.2(I), Proposition 4.1, and their ${}_AB(A^*)$-versions (see also Remark 3.3(b)), we can obtain the theorem below regarding commutants and bicommutants of module map algebras.

Theorem 4.2. *Let A be a Banach algebra. Then the following assertions hold.*

(i) *Suppose that A has a BRAI. Then we have $B_A(A^*)^c = {}_{A^{**}}B(A^*) = {}_AB^\sigma(A^*)^{cc}$. Moreover, ${}_AB^\sigma(A^*)^{cc} = {}_AB^\sigma(A^*) \iff A$ is left-right quotient strongly Arens irregular.*
(ii) *Suppose that A has a BLAI. Then we have ${}_AB(A^*)^c = B_{A^{**}}(A^*) = B_A^\sigma(A^*)^{cc}$. Moreover, $B_A^\sigma(A^*)^{cc} = B_A^\sigma(A^*) \iff A$ is right-left quotient strongly Arens irregular.*

In particular, if A is a commutative Banach algebra with a BAI, then

$$A \text{ is Q-SAI} \iff M(A)^{cc} = M(A),$$

where $M(A)$ is the multiplier algebra of A and $M(A) \hookrightarrow B(A^*)$.

The corollary below is immediate by Theorem 4.2, Corollary 3.8, and the ${}_AB(A^*)$-version of Corollary 3.8. It shows that for Banach algebras of type (M), left and right strong Arens irregularities are in fact equivalent to certain commutation relations.

Corollary 4.3. *Let A be a Banach algebra of type (M). Then we have*

(i) *A is LSAI* $\Longleftrightarrow B_A(A^*)^c = {}_AB^\sigma(A^*) \Longleftrightarrow {}_AB^\sigma(A^*)^{cc} = {}_AB^\sigma(A^*)$;
(ii) *A is RSAI* $\Longleftrightarrow {}_AB(A^*)^c = B^\sigma_A(A^*) \Longleftrightarrow B^\sigma_A(A^*)^{cc} = B^\sigma_A(A^*)$.

Recall that $L_1(G)$ is of type (M) and is SAI. Therefore, Corollary 4.3(i) improves and extends [12, Theorem 5.1], which shows that $B_{L_1(G)}(L_\infty(G))^c = \{\lambda^*_\mu\colon\ \mu \in M(G)\}$, where G is a locally compact group and $\lambda_\mu(f) = \mu * f$ $(f \in L_1(G))$.

For the canonical embedding $A \subseteq {}_AB^\sigma(A^*)$, we have $B_A(A^*) = A^c$. Similarly, ${}_AB(A^*) = A^c$ if A is embedded into $B^\sigma_A(A^*)$. Therefore, by Theorem 3.2(V), Proposition 4.1, and Corollary 4.3, we obtain the following two corollaries, where $LM(A)$ and $RM(A)$ are identified with their canonical images in $B(A^*)$.

Corollary 4.4. *Let A be a Banach algebra satisfying $\langle A^2\rangle = A$. Then we have*

(i) $A^c = LM(A)^c$ *via* $A \hookrightarrow LM(A)$;
(ii) $A^c = RM(A)^c$ *via* $A \hookrightarrow RM(A)$;
(iii) $RM(A) = LM(A)^c \Longleftrightarrow A \cdot A^{**} \subseteq A$.

Corollary 4.5. *Let A be a Banach algebra of type (M). Then we have*

(i) *A is LSAI* $\Longleftrightarrow LM(A)^{cc} = LM(A)$;
(ii) *A is RSAI* $\Longleftrightarrow RM(A)^{cc} = RM(A)$.

Therefore, for every unital weakly sequentially complete Banach algebra A, we have

(a) *A is LSAI* $\Longleftrightarrow A^{cc} = A$, *where* $A \cong LM(A)$;
(b) *A is RSAI* $\Longleftrightarrow A^{cc} = A$, *where* $A \cong RM(A)$.

We have the following immediate corollary of Theorem 4.2 and Corollary 4.5. For Fourier algebras of amenable locally compact groups, the implication (ii) $\Longrightarrow$ (i) below was shown in [32, Theorem 6.4].

Corollary 4.6. *Let A be a commutative Banach algebra of type (M). Then the following statements are equivalent*:

(i) *A is SAI*;
(ii) *A is Q-SAI*;
(iii) $M(A)^{cc} = M(A)$.

Let $\mathcal{M}$ be a von Neumann algebra standardly represented on a Hilbert space H with $\mathcal{M}_*$ a Banach algebra of type (M) (e.g., $\mathcal{M}_*$ is a unital Banach algebra or a separable Banach algebra with a BAI). It is interesting to compare certain commutation theorems over $CB(B(H))$, $B(\mathcal{M})$, $B(H)$, and $B(\mathcal{M}^*)$.

Firstly, on the one hand, we have in $CB(B(H))$ that

$$CB^\sigma_{\mathcal{M}}\big(B(H)\big)^c = CB_{\mathcal{M}'}\big(B(H)\big) \quad \text{and} \quad CB_{\mathcal{M}}\big(B(H)\big)^c = CB^\sigma_{\mathcal{M}'}\big(B(H)\big),$$

where $CB_{\mathcal{M}}(B(H))$ is the algebra of completely bounded $\mathcal{M}$-bimodule maps on $B(H)$, and $\mathcal{M}'$ is the commutant of $\mathcal{M}$ in $B(H)$ (cf. (2.7) and (2.8) in [41] and references therein). On the other hand, by Proposition 4.1 and Corollary 4.3, we have

$${}_{\mathcal{M}_*}B^{\sigma}(\mathcal{M})^c = B_{\mathcal{M}_*}(\mathcal{M}), \quad \text{and} \quad {}_{\mathcal{M}_*}B(\mathcal{M})^c = B^{\sigma}_{\mathcal{M}_*}(\mathcal{M}) \quad \Longleftrightarrow \quad \mathcal{M}_* \text{ is RSAI.}$$

Secondly, the von Neumann double commutation theorem says that $\mathcal{M}'' = \mathcal{M}$. However, when $\mathcal{M}_*$ is unital, for the canonical representation $\mathcal{M}_* \longrightarrow B_{\mathcal{M}_*}(\mathcal{M})$, the double commutation relation $(\mathcal{M}_*)^{cc} = \mathcal{M}_*$ holds precisely when $\mathcal{M}_*$ is RSAI, which is not true, for example, if $\mathcal{M} = VN(SU(3))$ (cf. [36]).

Finally, in contrast to the bicommutant theorem $\mathcal{M}'' = \mathcal{M}$ in $B(H)$, we have $[\mathcal{M}^{cc} = \mathcal{M}$ in $B(\mathcal{M}^*)] \Longleftrightarrow [\dim(\mathcal{M}) < \infty]$ due to Theorem 4.2(ii) and the fact that $\mathcal{M}$ is Arens regular.

We end this section with the following remark on measure algebras of locally compact groups.

Remark 4.7. For a locally compact group G, by Proposition 2.7(iv), we have

$$B^{w^*}_{M(G)}\big(M(G)^*\big) = \mathfrak{Z}_t\big(M(G)^{**}, \Box\big) \quad \text{and} \quad B_{M(G)^{**}}\big(M(G)^*\big) = \mathfrak{Z}_t\big(M(G)^{**}, \diamond\big).$$

It is known from [39, Theorem 3.5] that $M(G)$ is SAI if either G is non-compact with $|G|$ a non-measurable cardinal or $2^{\chi(G)} \leqslant \kappa(G)$, where $\kappa(G)$ and $\chi(G)$ denote, respectively, the compact covering number and the local weight of G. Recently, Losert, Neufang, Pachl and Steprāns [37] showed that $M(G)$ is SAI for all locally compact groups G. Therefore, by Corollary 4.5, we always have

$$B_{M(G)^{**}}\big(M(G)^*\big) = B^{w^*}_{M(G)}\big(M(G)^*\big) = M(G) = M(G)^{cc},$$

where $M(G) \cong B^{\sigma}_{M(G)}(M(G)^*)$ and the commutant is taken in $B(M(G)^*)$. Note that $M(G)^{cc} = M(G)$ also holds in $B(L_\infty(G))$ by Corollary 4.5. The reader is referred to [7] for other recent results on $M(G)$.

5. A new product for module maps and some applications

Let A be a Banach algebra. In [20], we introduced the closed subspace $\langle A^*A\rangle^*_R$ of $\langle A^*A\rangle^*$ given by

$$\langle A^*A\rangle^*_R = \big\{m \in \langle A^*A\rangle^*\colon\ \langle A^*A\rangle \diamond m \subseteq \langle A^*A\rangle\big\}. \tag{5.1}$$

For $m \in \langle A^*A\rangle^*_R$ and $n \in \langle A^*A\rangle^*$, the functional $m \diamond n \in \langle A^*A\rangle^*$ is naturally defined by

$$\langle f, m \diamond n\rangle = \langle f \diamond m, n\rangle \quad \big(f \in \langle A^*A\rangle\big).$$

Then $(\langle A^*A\rangle^*_R, \diamond)$ is a Banach algebra, and we showed [20, Theorem 2(i)] that

$$\mathfrak{Z}_t\big(\langle A^*A\rangle^*\big) = \big\{m \in \langle A^*A\rangle^*_R\colon\ m \,\Box\, n = m \diamond n \text{ for all } n \in \langle A^*A\rangle^*\big\}. \tag{5.2}$$

On the other hand, $(\langle A^*A\rangle^*_R, \diamond)$ is also a left topological semigroup under the $\sigma(\langle A^*A\rangle^*_R, \langle A^*A\rangle)$-topology. By definition, the topological centre $\mathfrak{Z}_t(\langle A^*A\rangle^*_R)$ of $(\langle A^*A\rangle^*_R, \diamond)$ is the set of all $m \in \langle A^*A\rangle^*_R$ such that the map $n \longmapsto n \diamond m$ is $\sigma(\langle A^*A\rangle^*_R, \langle A^*A\rangle)$-continuous. Since A is w^*-dense in $\langle A^*A\rangle^*$, we have

$$\mathfrak{Z}_t\big(\langle A^*A\rangle^*_R\big) = \big\{m \in \langle A^*A\rangle^*_R\colon\ n \diamond m = n \,\Box\, m \text{ for all } n \in \langle A^*A\rangle^*_R\big\}. \tag{5.3}$$

We note that, in general, the topological centre $\mathfrak{Z}_t(\langle A^*A\rangle^*_R)$ of $\langle A^*A\rangle^*_R$ and the auxiliary topological centre $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond$ of $\langle A^*A\rangle^*$ are not related. For example, if A is the group algebra $L_1(G)$ of a locally compact group G, then $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamond = \mathfrak{Z}_t(\langle A^*A\rangle^*) = M(G)$; however, $\delta_{e_G} \notin \mathfrak{Z}_t(\langle A^*A\rangle^*_R)$ unless G is an SIN group (cf. [20, Theorem 19]).

We consider below the new Banach algebra structure on $B_A(A^*)$ associated with $(\langle A^*A\rangle^*_R, \diamond)$. Note that, for each $T \in B_A(A^*)$, we have $T(\langle A^*A\rangle) \subseteq \langle A^*A\rangle$. Let

$$B_A\big(A^*\big)_R = \big\{T \in B_A\big(A^*\big)\colon\ \langle A^*A\rangle \diamond T^*(A) \subseteq \langle A^*A\rangle\big\}. \tag{5.4}$$

Then $B_A(A^*)_R$ is a norm closed subspace of $B_A(A^*)$. Notice that we have $A^* \diamond \mathfrak{Z}_t(A^{**}, \Box) \subseteq \langle A^*A\rangle$ (cf. [6, Proposition 2.20]). Therefore, due to (2.11) and (2.12), we have

$$B_A^{l,w^*}\big(A^*\big) \subseteq \big\{T \in B_A\big(A^*\big)\colon\ T^*(A)|_{\langle A^*A\rangle^*} \subseteq \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big)\big\} \subseteq B_A\big(A^*\big)_R \subseteq B_A\big(A^*\big). \tag{5.5}$$

Proposition 5.1. *Let A be a Banach algebra and let $\Phi : \langle A^*A\rangle^* \longrightarrow B_A(A^*)$ be the canonical representation of $\langle A^*A\rangle^*$. Then the following assertions hold.*

(i) *$\Phi(\langle A^*A\rangle^*_R) \subseteq B_A(A^*)_R$, and the equality holds if A has a BRAI.*
(ii) *If $\langle A^*A\rangle$ is two-sided introverted in A^*, then $B_A(A^*) = B_A(A^*)_R$; the converse holds if $\langle A^2\rangle = A$.*

Proof. (i) This is true, since $(A^*A) \diamond (m_L)^*(A) = (A^*A^2) \diamond m$.

(ii) Suppose that $\langle A^*A\rangle$ is two-sided introverted in A^*. Then we have $\langle A^*A\rangle \diamond A^{**} \subseteq \langle A^*A\rangle$. Therefore, $B_A(A^*) = B_A(A^*)_R$. Conversely, suppose that $B_A(A^*) = B_A(A^*)_R$ and $\langle A^2\rangle = A$. If $m \in \langle A^*A\rangle^*$, then $m_L \in B_A(A^*)_R$, and thus $(A^*A^2) \diamond m = (A^*A) \diamond (m_L)^*(A) \subseteq \langle A^*A\rangle$. Therefore, $\langle A^*A\rangle \diamond m \subseteq \langle A^*A\rangle$ for all $m \in \langle A^*A\rangle^*$, since $\langle A^2\rangle = A$; that is, $\langle A^*A\rangle$ is right and hence two-sided introverted in A^*. $\Box$

In the following, we assume that A has a contractive BRAI. Then $(A^{**}, \Box)$ has a right identity E of norm 1, and $(\langle A^*A\rangle^*, \Box) \cong (B_A(A^*), \circ)$ via the isometric and w^*-w^* homeomorphic algebra isomorphism Φ (cf. Proposition 2.6). For $T \in B_A(A^*)_R$ and $S \in B_A(A^*)$, we define

$$\big\langle (T \diamond S)(f), a\big\rangle = \big\langle f, T^*(a) \diamond S^*(E)\big\rangle = \big\langle f \diamond T^*(a), S^*(E)\big\rangle \quad \big(f \in A^*,\ a \in A\big). \tag{5.6}$$

It is easily seen that $T \diamond S \in B_A(A^*)$ with $\|T \diamond S\| \leqslant \|T\|\|S\|$, and $T \diamond S$ is independent of the choice of E, since

$$A^* \diamond T^*(A) = A^* \diamond T^*\big(A^2\big) \subseteq \big\langle A^*A\big\rangle \diamond T^*(A) \subseteq \big\langle A^*A\big\rangle.$$

Let $T_1, T_2 \in B_A(A^*)_R$, $f \in A^*$, and $a \in A$. Since $f \diamondsuit T_1^*(a) \in \langle A^*A \rangle = A^*A$, we have $f \diamondsuit T_1^*(a) = g \cdot b$ for some $g \in A^*$ and $b \in A$. Then we obtain

$$f \diamondsuit (T_1 \diamondsuit T_2)^*(a) = \big(f \diamondsuit T_1^*(a)\big) \diamondsuit T_2^*(E) = (g \cdot b) \diamondsuit T_2^*(E) = g \diamondsuit T_2^*(b) \in \langle A^*A \rangle.$$

It follows that $T_1 \diamondsuit T_2 \in B_A(A^*)_R$. Also, we have $(T_1 \diamondsuit T_2) \diamondsuit S = T_1 \diamondsuit (T_2 \diamondsuit S)$ $(S \in B_A(A^*))$. Therefore, $(B_A(A^*)_R, \diamondsuit)$ is an associative Banach algebra, and $(\langle A^*A \rangle_R^*, \diamondsuit) \cong (B_A(A^*)_R, \diamondsuit)$ via the isometric algebra isomorphism $\Phi|_{\langle A^*A \rangle_R^*}$ (cf. Proposition 5.1(i)).

Since $(B_A(A^*)_R, \diamondsuit)$ is a left topological semigroup under the relative w^*ot-topology, its topological centre $\mathfrak{Z}_t(B_A(A^*)_R)$ is defined. In fact, we have $\mathfrak{Z}_t(B_A(A^*)_R) = \Phi(\mathfrak{Z}_t(\langle A^*A \rangle_R^*))$. Therefore, $\mathfrak{Z}_t(B_A(A^*)_R)$ is a norm closed subalgebra of $(B_A(A^*), \circ)$ and of $(B_A(A^*)_R, \diamondsuit)$, and

$$\mathfrak{Z}_t\big(B_A\big(A^*\big)_R\big) = \big\{T \in B_A\big(A^*\big)_R \colon\ S \diamondsuit T = S \circ T \text{ for all } S \in B_A\big(A^*\big)_R\big\}. \tag{5.7}$$

Summarizing the above, we have the following theorem on the Banach algebra $(B_A(A^*)_R, \diamondsuit)$.

Theorem 5.2. *Let A be a Banach algebra with a contractive BRAI. Then the canonical representation $\Phi : \langle A^*A \rangle^* \longrightarrow B_A(A^*)$ induces the isometric algebra isomorphisms*

$$\big(B_A\big(A^*\big)_R, \diamondsuit\big) \cong \big(\langle A^*A \rangle_R^*, \diamondsuit\big) \quad \textit{and} \quad \mathfrak{Z}_t\big(B_A\big(A^*\big)_R\big) \cong \mathfrak{Z}_t\big(\langle A^*A \rangle_R^*\big).$$

Clearly, id is the identity of $(B_A(A^*), \circ)$, $id \in B_A(A^*)_R$, and $id \diamondsuit S = S$ for all $S \in B_A(A^*)$. It is also easy to show that $S \diamondsuit id$ is the w^*-limit of $(S \circ (e_\alpha)_L)$ in $B_A(A^*)$ if $S \in B_A(A^*)_R$, where (e_α) is a contractive BRAI of A.

Corollary 5.3. *Let A be a Banach algebra with a contractive BRAI. Then the following statements are equivalent*:

(i) $B_A^\sigma(A^*) \subseteq \mathfrak{Z}_t(B_A(A^*)_R)$;
(ii) $id \in \mathfrak{Z}_t(B_A(A^*)_R)$;
(iii) *id is the identity of* $(B_A(A^*)_R, \diamondsuit)$;
(iv) $\langle A^*A \rangle = \langle AA^*A \rangle$.

Proof. Obviously, we have (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). The equivalence (iii) $\Longleftrightarrow$ (iv) holds by Theorem 5.2 and [20, Theorem 10]. The implication (iii) $\Longrightarrow$ (i) follows from Theorem 5.2 and [20, Lemma 9(ii)], which shows that $RM(A) \subseteq \mathfrak{Z}_t(\langle A^*A \rangle_R^*)$ if $(\langle A^*A \rangle_R^*, \diamondsuit)$ is unital. □

Let $WAP(A)$ be the space of weakly almost periodic functionals on A. That is, $WAP(A)$ is the subspace of A^* consisting of all $f \in A^*$ such that the map $A \longrightarrow A^*$, $a \longmapsto f \cdot a$ is weakly compact. Notice that the composition of the canonical embedding $A \longrightarrow A^{**}$ and the adjoint of the above map is the map $A \longrightarrow A^*$, $a \longmapsto a \cdot f$. Therefore, if $f \in A^*$, then $f \in WAP(A)$ if and only if $A \longrightarrow A^*$, $a \longmapsto a \cdot f$ is weakly compact. It is easy to show that $WAP(A) \subseteq \langle A^*A \rangle$ if A has a BLAI, and $WAP(A) \subseteq \langle AA^* \rangle$ if A has a BRAI (cf. the proof of [6, Proposition 3.12]). The following theorem links the module map property described in Theorem 3.2(IV) to the spaces $WAP(A)$ and $\langle A^*A \rangle_R^*$. This in particular generalizes [34, Theorem 3.6], where the equivalence between (i)–(iii) below was shown for A with a BAI.

Theorem 5.4. *Let A be a Banach algebra satisfying $\langle A^2\rangle = A$. Then the following statements are equivalent*:

(i) $\langle A^*A\rangle \subseteq WAP(A)$;
(ii) $\langle A^*A\rangle^* = \mathfrak{Z}_t(\langle A^*A\rangle^*)$;
(iii) $A \cdot A^{**} \subseteq \mathfrak{Z}_t(A^{**}, \Box)$;
(iv) $A \cdot \langle A^*A\rangle^* \subseteq \mathfrak{Z}_t(\langle A^*A\rangle^*)$;
(v) $B_A(A^*) = B_A^{l,w^*}(A^*)$.

In addition, if A is an involutive Banach algebra such that $\langle A^*A\rangle^*$ is left faithful (*e.g., A has a BRAI or is LQ-SAI, or $\langle A^*A\rangle = \langle AA^*A\rangle$*), *then* (i)–(v) *are equivalent to*

(vi) $\langle A^*A\rangle_R^* = \mathfrak{Z}_t(\langle A^*A\rangle^*)$.

Furthermore, if A is an involutive Banach algebra with a contractive BRAI, then the above (i)–(vi) *are also equivalent to $B_A(A^*)_R = B_A^{w^*}(A^*)$.*

Proof. It is known that $\langle A^*A\rangle \subseteq WAP(A)$ if and only if $\langle A^*A\rangle$ is two-sided introverted in A^* and $\langle x, m\Box n\rangle = \langle x, m\Diamond n\rangle$ for all $x \in \langle A^*A\rangle$ and $m, n \in A^{**}$ (cf. [6, Propositions 3.11 and 5.7]). Due to (5.2), we have (i) $\Longleftrightarrow$ (ii). The equivalence between (ii)–(v) has been proved in Theorem 3.2(IV).

Obviously, (ii) $\Longrightarrow$ (vi) holds. Suppose that A is an involutive Banach algebra such that $\langle A^*A\rangle^*$ is left faithful, which is the case, by Proposition 3.15(e), if A has a BRAI or is LQ-SAI, or $\langle A^*A\rangle = \langle AA^*A\rangle$. By Proposition 3.15(b), we have $\mathfrak{Z}_t(\langle A^*A\rangle^*) \cap \langle AA^*A\rangle^{\perp}_{\langle A^*A\rangle} = \{0\}$. To show (vi) $\Longrightarrow$ (ii), we suppose that $\langle A^*A\rangle_R^* = \mathfrak{Z}_t(\langle A^*A\rangle^*)$. It is evident that $\langle AA^*A\rangle^{\perp}_{\langle A^*A\rangle} \subseteq \langle A^*A\rangle_R^*$, and thus $\langle AA^*A\rangle^{\perp}_{\langle A^*A\rangle} = \{0\}$, or equivalently, $\langle A^*A\rangle = \langle AA^*A\rangle$. According to [20, Proposition 11(v)], the space $\langle A^*A\rangle$ is two-sided introverted in A^*. It follows that $\langle A^*A\rangle^* = \langle A^*A\rangle_R^* = \mathfrak{Z}_t(\langle A^*A\rangle^*)$. Therefore, we have (vi) $\Longrightarrow$ (ii). The final assertion follows from Theorem 5.2 and Proposition 2.7. $\Box$

6. Quotient strong Arens irregularity over locally compact quantum groups

Let us start this section with a brief recalling of some notation related to locally compact quantum groups. The reader is referred to [24,25] by Kustermans and Vaes for more information. See also [20,21,23]. Let $\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \varphi, \psi)$ be a von Neumann algebraic locally compact quantum group. Then the pre-adjoint of the co-multiplication Γ induces on $L_1(\mathbb{G}) = L_\infty(\mathbb{G})_*$ an associative multiplication $\star$ such that $L_1(\mathbb{G})$ is a faithful completely contractive involutive Banach algebra satisfying $\langle L_1(\mathbb{G}) \star L_1(\mathbb{G})\rangle = L_1(\mathbb{G})$. In the case where $L_\infty(\mathbb{G})$ is $L_\infty(G)$ or $VN(G)$ for a locally compact group G, the algebra $L_1(\mathbb{G})$ is the usual convolution group algebra $L_1(G)$, respectively, the Fourier algebra $A(G)$. Let $C_0(\mathbb{G})$ be the reduced C^*-algebra of $\mathbb{G}$, let $M(C_0(\mathbb{G}))$ be the multiplier algebra of $C_0(\mathbb{G})$, and let $LUC(\mathbb{G})$ be the subspace $\langle L_1(\mathbb{G})^* \star L_1(\mathbb{G})\rangle$ of $L_\infty(\mathbb{G})$. It is known from [43] that

$$C_0(\mathbb{G}) \subseteq LUC(\mathbb{G}) \subseteq M\big(C_0(\mathbb{G})\big) \subseteq L_\infty(\mathbb{G}). \tag{6.1}$$

The space $M(\mathbb{G}) = C_0(\mathbb{G})^*$ is also a faithful completely contractive involutive Banach algebra with a multiplication induced by Γ, and $L_1(\mathbb{G})$ can be identified with a closed two-sided ideal in

$M(\mathbb{G})$ via the restriction map $f \longmapsto f|_{C_0(\mathbb{G})}$. When $L_\infty(\mathbb{G}) = L_\infty(G)$ (respectively, $VN(G)$), $M(\mathbb{G})$ is the measure algebra $M(G)$ of G (respectively, the reduced Fourier–Stieltjes algebra $B_\lambda(G)$ of G).

Now we let X be a left introverted subspace of $L_\infty(\mathbb{G})$ such that $C_0(\mathbb{G}) \subseteq X \subseteq M(C_0(\mathbb{G}))$. Let $M(\widetilde{C_0(\mathbb{G})})$ be the idealizer of $C_0(\mathbb{G})$ in $C_0(\mathbb{G})^{**}$. That is,

$$M\big(\widetilde{C_0(\mathbb{G})}\big) = \big\{x \in C_0(\mathbb{G})^{**}\colon\ \widetilde{a}x, x\widetilde{a} \in C_0(\mathbb{G}) \text{ for all } a \in C_0(\mathbb{G})\big\}, \tag{6.2}$$

where $\widetilde{a}$ is the canonical image of a in $C_0(\mathbb{G})^{**}$. It is well known that we have a C^*-algebra isomorphism $M(C_0(\mathbb{G})) \cong M(\widetilde{C_0(\mathbb{G})})$, extending the canonical identification $C_0(\mathbb{G}) \cong \widetilde{C_0(\mathbb{G})}$. This yields a complete isometry

$$\tau : X \subseteq M\big(C_0(\mathbb{G})\big) \cong M\big(\widetilde{C_0(\mathbb{G})}\big) \subseteq C_0(\mathbb{G})^{**} \tag{6.3}$$

satisfying $\tau(C_0(\mathbb{G})) = \widetilde{C_0(\mathbb{G})}$. Correspondingly, we let $\widetilde{X} = \tau(X) \subseteq M(\mathbb{G})^*$. Then $M(\mathbb{G})$ is isometrically identified with a subspace of $\widetilde{X}^*$ via $\mu \longmapsto \mu|_{\widetilde{X}}$ since $\widetilde{C_0(\mathbb{G})} \subseteq \widetilde{X}$. It was shown in [22, Proposition 2.1] that there exists a completely isometric algebra homomorphism

$$\pi : M(\mathbb{G}) \longrightarrow X^* \tag{6.4}$$

which is an $L_1(\mathbb{G})$-module and $C_0(\mathbb{G})$-module map such that $\pi^*|_X = \tau$. Clearly, X is always a left $M(\mathbb{G})$-submodule of $L_\infty(\mathbb{G})$. Therefore, $\tau : X \longrightarrow M(\mathbb{G})^*$ is an $L_1(\mathbb{G})$-module map, and is an $M(\mathbb{G})$-module map if X is also a right $M(\mathbb{G})$-submodule of $L_\infty(\mathbb{G})$ (cf. [22, Corollary 2.4]).

Proposition 6.1. *Let $\mathbb{G}$ be a locally compact quantum group and let X be a left introverted subspace of $L_\infty(\mathbb{G})$ such that $C_0(\mathbb{G}) \subseteq X \subseteq M(C_0(\mathbb{G}))$. Then the following assertions hold.*

(i) *$\widetilde{X}$ is an $L_1(\mathbb{G})$-submodule of $M(\mathbb{G})^*$ and $\tau : X \longrightarrow \widetilde{X}$ is a completely isometric $L_1(\mathbb{G})$-module isomorphism.*
(ii) *If X is a right $M(\mathbb{G})$-submodule of $L_\infty(\mathbb{G})$, then $\widetilde{X}$ is an $M(\mathbb{G})$-submodule of $M(\mathbb{G})^*$ and $\tau : X \longrightarrow \widetilde{X}$ is an $M(\mathbb{G})$-module map.*
(iii) *If $X = \langle X \star L_1(\mathbb{G})\rangle$, then $\widetilde{X}$ is left introverted in $M(\mathbb{G})^*$ and $\tau^* : \widetilde{X}^* \longrightarrow X^*$ is a completely isometric algebra isomorphism and an $M(\mathbb{G})$-module map. In this case, we have*

$$\widetilde{X}^* = M(\mathbb{G}) \oplus \widetilde{C_0(\mathbb{G})}^{\perp} \quad \textit{and} \quad \tau^*\big(M(\mathbb{G})\big) = \pi\big(M(\mathbb{G})\big) \subseteq \mathfrak{Z}_t\big(X^*\big) = \tau^*\big(\mathfrak{Z}_t\big(\widetilde{X}^*\big)\big),$$

where $\widetilde{C_0(\mathbb{G})}^{\perp} = \{\widetilde{m} \in \widetilde{X}^*\colon\ \widetilde{m}|_{\widetilde{C_0(\mathbb{G})}} = 0\}$.

Proof. (i) and (ii). This is clear by the above discussions.

(iii) Suppose that $X = \langle X \star L_1(\mathbb{G})\rangle$. Then X is an $M(\mathbb{G})$-submodule of $L_\infty(\mathbb{G})$ and hence, by (i) and (ii), $\widetilde{X}$ is an $M(\mathbb{G})$-submodule of $M(\mathbb{G})^*$ and both τ and τ^* are completely isometric $M(\mathbb{G})$-module maps. Let $\widetilde{m} \in \widetilde{X}^*$, $\widetilde{x} = \tau(x) \in \widetilde{X}$ with $x \in X$, and $\mu \in M(\mathbb{G})$. Then $m = \tau^*(\widetilde{m}) \in X^*$, and hence $m \,\Box\, x \in X$. By [22, Proposition 2.1(i)] and noticing that $X = \langle X \star L_1(\mathbb{G})\rangle$, we have $\langle \pi(\mu) \,\Box\, f, x\rangle = \langle f, x \star \mu\rangle$ for all $f \in L_1(\mathbb{G})$. Since $L_1(\mathbb{G})$ is w^*-dense in X^*, there exists a net (f_i) in $L_1(\mathbb{G})$ such that $f_i \longrightarrow m$ in the weak*-topology on X^*. We obtain that $\pi(\mu) \Box f_i \longrightarrow \pi(\mu) \Box m$ in the weak*-topology on X^*, since $\pi(\mu) \in \mathfrak{Z}_t(X^*)$ (cf. [22, Proposition 2.1(iii)]). Then we have

$$\big\langle \pi(\mu) \Box m, x \big\rangle = \lim_i \big\langle \pi(\mu) \Box f_i, x \big\rangle = \lim_i \langle f_i, x \star \mu \rangle = \langle m, x \star \mu \rangle.$$

On the other hand, $\langle \pi(\mu) \Box m, x \rangle = \langle \pi(\mu), m \Box x \rangle = \langle \tau(m \Box x), \mu \rangle$. Therefore, we have

$$\langle \widetilde{m} \Box \widetilde{x}, \mu \rangle = \big\langle \widetilde{m}, \tau(x) \star \mu \big\rangle = \big\langle \widetilde{m}, \tau(x \star \mu) \big\rangle = \langle m, x \star \mu \rangle = \big\langle \pi(\mu) \Box m, x \big\rangle = \big\langle \tau(m \Box x), \mu \big\rangle.$$

Thus $\widetilde{m} \Box \widetilde{x} = \tau(m \Box x) \in \tau(X) = \widetilde{X}$. That is,

$$\widetilde{m} \Box \tau(x) = \tau\big(\tau^*(\widetilde{m}) \Box x\big) \in \widetilde{X} \quad \big(\widetilde{m} \in \widetilde{X}^*,\ x \in X\big). \tag{6.5}$$

It follows from (6.5) that $\widetilde{X}$ is left introverted in $M(\mathbb{G})^*$ and $\tau^* : \widetilde{X}^* \longrightarrow X^*$ is an algebra isomorphism. It is obvious that the final assertion holds, noticing that $\pi(M(\mathbb{G})) \subseteq \mathfrak{Z}_t(X^*)$ (cf. [22, Proposition 2.1]). $\Box$

By definition, the space $LUC(\mathbb{G})$ is a left introverted $M(\mathbb{G})$-submodule of $L_\infty(\mathbb{G})$ satisfying $LUC(\mathbb{G}) = \langle LUC(\mathbb{G}) \star L_1(\mathbb{G}) \rangle$. Therefore, $\widetilde{LUC(\mathbb{G})}$ is left introverted in $M(\mathbb{G})^*$. As we shall show, the space $\widetilde{LUC(\mathbb{G})}$ can be obtained naturally as a space of certain left uniformly continuous functionals on $M(\mathbb{G})$. The equality given in (6.6) below is a quantum group version of [11, Lemma 2.18] on the classical space $LUC(G)$ of left uniformly continuous functions on a locally compact group G.

Theorem 6.2. *Let $\mathbb{G}$ be a locally compact quantum group and let τ be the map given in* (6.3) *for $X = LUC(\mathbb{G})$. Then $\tau : LUC(\mathbb{G}) \longrightarrow C_0(\mathbb{G})^{**}$ is a completely isometric $M(\mathbb{G})$-module map and we have*

$$\widetilde{LUC(\mathbb{G})} = \big\langle M(\mathbb{G})^* \star L_1(\mathbb{G}) \big\rangle. \tag{6.6}$$

Therefore, $\langle M(\mathbb{G})^* \star L_1(\mathbb{G}) \rangle$ is a unital C^*-subalgebra of $C_0(\mathbb{G})^{**}$ if $\mathbb{G}$ is semi-regular.

Proof. We only need show that $\langle M(\mathbb{G})^* \star L_1(\mathbb{G}) \rangle \subseteq \widetilde{LUC(\mathbb{G})}$. Let $y \in M(\mathbb{G})^*$, $f, g \in L_1(\mathbb{G})$, and $\mu \in M(\mathbb{G})$. Let $x = y|_{L_1(\mathbb{G})} \in L_\infty(\mathbb{G})$. Then we have

$$\big\langle y \star (f \star g), \mu \big\rangle = \big\langle y, f \star (g \star \mu) \big\rangle = \big\langle x, f \star (g \star \mu) \big\rangle = \langle x \star f, g \star \mu \rangle.$$

Since $x \star f \in LUC(\mathbb{G})$ and $g \star \mu \in L_1(\mathbb{G})$, by [22, Proposition 2.1], we have

$$\langle x \star f, g \star \mu \rangle = \big\langle x \star f, \pi(g \star \mu) \big\rangle = \big\langle \tau(x \star f), g \star \mu \big\rangle = \big\langle \tau(x \star f) \star g, \mu \big\rangle = \big\langle \tau\big(x \star (f \star g)\big), \mu \big\rangle.$$

It follows that $y \star (f \star g) = \tau(x \star (f \star g)) \in \tau(LUC(\mathbb{G}))$. Therefore, we have $\langle M(\mathbb{G})^* \star L_1(\mathbb{G}) \rangle \subseteq \tau(LUC(\mathbb{G})) = \widetilde{LUC(\mathbb{G})}$, since $\langle L_1(\mathbb{G}) \star L_1(\mathbb{G}) \rangle = L_1(\mathbb{G})$. The final assertion holds by [23, Theorem 5.6]. $\Box$

Remark 6.3. Note that the adjoint of the inclusion map $L_1(\mathbb{G}) \longrightarrow M(\mathbb{G})$ is the normal and surjective $*$-homomorphism $C_0(\mathbb{G})^{**} \longrightarrow L_\infty(\mathbb{G})$, $x \longmapsto x|_{L_1(\mathbb{G})}$, which extends the inclusion $C_0(\mathbb{G}) \subseteq L_\infty(\mathbb{G})$. The kernel of this $*$-homomorphism is the w^*-closed ideal $L_1(\mathbb{G})^\perp$ in $C_0(\mathbb{G})^{**}$. Then there exists a central projection p in $C_0(\mathbb{G})^{**}$ such that $L_1(\mathbb{G})^\perp = (1 - p)C_0(\mathbb{G})^{**}$, and thus we have

$$C_0(\mathbb{G})^{**} = pC_0(\mathbb{G})^{**} \oplus_\infty L_1(\mathbb{G})^\perp \cong L_\infty(\mathbb{G}) \oplus_\infty L_1(\mathbb{G})^\perp \quad \text{via } x \oplus y \longmapsto x|_{L_1(\mathbb{G})} \oplus y.$$

Let $\kappa : L_\infty(\mathbb{G}) \longrightarrow C_0(\mathbb{G})^{**}$ be the induced normal and injective $*$-homomorphism. By [22, Proposition 3.6], $L_1(\mathbb{G}) = M(\mathbb{G})$ if and only if $\kappa(C_0(\mathbb{G})) = \widetilde{C_0(\mathbb{G})}$ (respectively, $\kappa(M(C_0(\mathbb{G}))) = M(\widetilde{C_0(\mathbb{G})})$). It follows that the following statements are equivalent:

(i) $L_1(\mathbb{G}) = M(\mathbb{G})$;
(ii) $\kappa(LUC(\mathbb{G})) = \widetilde{LUC(\mathbb{G})}$;
(iii) $\kappa(L_\infty(\mathbb{G})) = C_0(\mathbb{G})^{**}$.

Therefore, we do not have $\kappa(LUC(\mathbb{G})) = \langle M(\mathbb{G})^* \star L_1(\mathbb{G})\rangle$ in general.

It is known from [23, Proposition 6.2] that $L_1(\mathbb{G})$ is Q-SAI if and only if the embedding $\pi : M(\mathbb{G}) \longrightarrow LUC(\mathbb{G})^*$ maps $M(\mathbb{G})$ onto $\mathfrak{Z}_t(LUC(\mathbb{G})^*)$. Note that, by the definition of the strong topological centre $\mathfrak{S}\mathfrak{Z}_t(LUC(\mathbb{G})^*)$ of $LUC(\mathbb{G})^*$ (cf. (3.5)) and the decomposition $LUC(\mathbb{G})^* = \pi(M(\mathbb{G})) \oplus C_0(\mathbb{G})^\perp$, we have

$$\mathfrak{S}\mathfrak{Z}_t\big(LUC(\mathbb{G})^*\big) = \pi\big(M(\mathbb{G})\big). \tag{6.7}$$

Let $\bullet$ denote the canonical $C_0(\mathbb{G})$-module actions on $M(\mathbb{G})$ and $L_\infty(\mathbb{G})$. Since $M(\mathbb{G}) = M(\mathbb{G}) \bullet C_0(\mathbb{G}) = C_0(\mathbb{G}) \bullet M(\mathbb{G})$ and $\pi : M(\mathbb{G}) \longrightarrow LUC(\mathbb{G})^*$ is a $C_0(\mathbb{G})$-module map, we obtain that

$$\begin{aligned} C_0(\mathbb{G}) \bullet LUC(\mathbb{G})^* \bullet C_0(\mathbb{G}) &= LUC(\mathbb{G})^* \bullet C_0(\mathbb{G}) = C_0(\mathbb{G}) \bullet LUC(\mathbb{G})^* \\ &= \pi\big(M(\mathbb{G})\big) \subseteq \mathfrak{Z}_t\big(LUC(\mathbb{G})^*\big). \end{aligned} \tag{6.8}$$

Therefore, we can characterize the quotient strong Arens irregularity of $L_1(\mathbb{G})$ in terms of the strong topological centre $\mathfrak{S}\mathfrak{Z}_t(LUC(\mathbb{G})^*)$ of $LUC(\mathbb{G})^*$ and the canonical $C_0(\mathbb{G})$-module structure on $LUC(\mathbb{G})^*$.

Proposition 6.4. *For any locally compact quantum group $\mathbb{G}$, the following statements are equivalent*:

(i) $L_1(\mathbb{G})$ *is Q-SAI*;
(ii) $\mathfrak{Z}_t(LUC(\mathbb{G})^*) = \mathfrak{S}\mathfrak{Z}_t(LUC(\mathbb{G})^*)$;
(iii) $\mathfrak{Z}_t(LUC(\mathbb{G})^*) = C_0(\mathbb{G}) \bullet LUC(\mathbb{G})^* \bullet C_0(\mathbb{G})$.

For a general Banach algebra A, it is known from (2.12) that A is LSAI $\Longrightarrow$ A is LQ-SAI. Similarly, we have A is RSAI $\Longrightarrow$ A is RQ-SAI. Therefore, for every locally compact quantum group $\mathbb{G}$, we have

$$M(\mathbb{G}) \text{ is SAI} \implies M(\mathbb{G}) \text{ is Q-SAI}, \quad \text{and} \quad L_1(\mathbb{G}) \text{ is SAI} \implies L_1(\mathbb{G}) \text{ is Q-SAI}.$$

Note that in general an SAI Banach algebra A (e.g., $A = L_1(\mathbb{T})$) can even have an infinite dimensional closed ideal which is Arens regular (cf. [19] and references therein). For the quantum measure algebra $M(\mathbb{G})$ and its closed ideal $L_1(\mathbb{G})$, however, we have the following theorem.

Theorem 6.5. *Let $\mathbb{G}$ be a locally compact quantum group. Then we have*

$$M(\mathbb{G}) \textit{ is Q-SAI} \implies L_1(\mathbb{G}) \textit{ is Q-SAI}.$$

Proof. Suppose that $M(\mathbb{G})$ is Q-SAI. We only need prove that $\mathfrak{Z}_t(\widetilde{LUC(\mathbb{G})}^*) \subseteq M(\mathbb{G})$ due to Proposition 6.1(iii). Let $\widetilde{m} \in \mathfrak{Z}_t(\widetilde{LUC(\mathbb{G})}^*)$. We show below that $\widetilde{m} \in M(\mathbb{G})$.

Let $f \in L_1(\mathbb{G})$. By Theorem 6.2, $f \star' \widetilde{m}$ is a well-defined element of $M(\mathbb{G})^{**}$ if we let $\langle f \star' \widetilde{m}, y\rangle = \langle \widetilde{m}, y \star f\rangle$ $(y \in M(\mathbb{G})^*)$. For all $p \in M(\mathbb{G})^{**}$ and $y \in M(\mathbb{G})^*$, we have

$$\begin{aligned}\big\langle \big(f \star' \widetilde{m}\big) \square\, p, y\big\rangle &= \big\langle f \star' \widetilde{m}, p \,\square\, y\big\rangle = \big\langle \widetilde{m}, (p \,\square\, y) \star f\big\rangle = \big\langle \widetilde{m}, p \,\square\, (y \star f)\big\rangle \\ &= \big\langle \widetilde{m} \,\square\, (p|_{\widetilde{LUC(\mathbb{G})}}), y \star f\big\rangle.\end{aligned}$$

Thus we obtain that $f \star' \widetilde{m} \in \mathfrak{Z}_t(M(\mathbb{G})^{**}, \square)$. Let $q = (f \star' \widetilde{m})|_{\langle M(\mathbb{G})^* \star M(\mathbb{G})\rangle}$. Then $q \in \mathfrak{Z}_t(\langle M(\mathbb{G})^* \star M(\mathbb{G})\rangle^*)$ (cf. Proposition 2.2(i)). Let $\mu \in M(\mathbb{G})$. By the assumption, we have $\mu_q = \mu \star q \in M(\mathbb{G})$. In particular, for all $\widetilde{x} \in \widetilde{LUC(\mathbb{G})}$, we have

$$\langle \mu_q, \widetilde{x}\rangle = \langle \mu \star q, \widetilde{x}\rangle = \langle q, \widetilde{x} \star \mu\rangle = \big\langle f \star' \widetilde{m}, \widetilde{x} \star \mu\big\rangle = \big\langle \widetilde{m}, \widetilde{x} \star (\mu \star f)\big\rangle = \big\langle (\mu \star f) \star \widetilde{m}, \widetilde{x}\big\rangle.$$

That is, we also have $(\mu \star f) \star \widetilde{m} = \mu_q \in M(\mathbb{G})$ in $\widetilde{LUC(\mathbb{G})}^*$. On the other hand, by Proposition 6.1(iii), $\widetilde{m} = \nu + \widetilde{n}$ for some $\nu \in M(\mathbb{G})$ and $\widetilde{n} \in \widetilde{C_0(\mathbb{G})}^{\perp}$. Then we have

$$(\mu \star f) \star \widetilde{m} = (\mu \star f) \star \nu + (\mu \star f) \star \widetilde{n} = \mu_q \in M(\mathbb{G}).$$

Notice that $L_1(\mathbb{G})$ is an ideal in $M(\mathbb{G})$ and $L_1(\mathbb{G}) \star \widetilde{C_0(\mathbb{G})}^{\perp} \subseteq \widetilde{C_0(\mathbb{G})}^{\perp}$. Therefore, $(\mu \star f) \star \widetilde{n} = 0$ for all $f \in L_1(\mathbb{G})$ and $\mu \in M(\mathbb{G})$. It follows that we have $\widetilde{n} = 0$, since $\widetilde{LUC(\mathbb{G})} = \langle \widetilde{LUC(\mathbb{G})} \star (M(\mathbb{G}) \star L_1(\mathbb{G}))\rangle$. Consequently, we have $\widetilde{m} = \nu \in M(\mathbb{G})$. □

The corollary below is immediate by Theorem 6.5, [20, Corollary 32], and Corollary 4.5.

Corollary 6.6. *Let $\mathbb{G}$ be a locally compact quantum group such that $L_1(\mathbb{G})$ is of type (M) with a central BAI. Then we have*

$$M(\mathbb{G}) \textit{ is SAI} \implies L_1(\mathbb{G}) \textit{ is SAI}.$$

Equivalently, we have $[M(\mathbb{G})^{cc} = M(\mathbb{G})$ in $B(M(\mathbb{G})^*)] \implies [M(\mathbb{G})^{cc} = M(\mathbb{G})$ in $B(L_1(\mathbb{G})^*)]$.

Therefore, for every amenable locally compact group G, we have $[B(G)$ is SAI$] \implies [A(G)$ is SAI$]$, or equivalently, $[B(G)^{cc} = B(G)$ in $B(W^*(G))] \implies [B(G)^{cc} = B(G)$ in $B(VN(G))]$.

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